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STOCHASTIC THEORY OF COMPOSITE MATERIALS WITH RANDOM WAVINESS OF THE REINFORCEMENTS

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Abstract—A general stochastic theory of the elastic properties of composite materials with continuous randomly curved spatial reinforcement is developed. The theory of random functions is utilized to evaluate the probabilistic characteristics of the local waviness of the reinforcement. A probabilistic extension of the orientation averaging model is developed to evaluate the elastic response of composites with multidirectional reinforcement having stochastic waviness. One fundamental advantage of the developed theory, compared to existing analytical approaches, is that an exact description of the reinforcement waviness is not required for predicting elastic properties. The only essential characteristics used as input data are the mean reinforcement paths and standard deviation of the local tangent, which is a random value characterizing the reinforcement path deflection from the "perfect" one.

It is shown that existing approaches for evaluating elastic response of the composite with imperfect continuous fiber reinforcement can be obtained from the developed theory as particular cases. The theory is illustrated with examples of a unidirectional composite and a helically wound composite with randomly curved reinforcements. Numerical examples show that even small local waviness of the reinforcement paths may significantly affect the elastic response of composites considered. © 1998 Elsevier Science Ltd. All rights reserved.

NOMENCLATURE

$\mathbf{a}^{(i)}$	vector $= \mathbf{e}'_i \,|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle}$
$\hat{\mathbf{b}}^{(i)}$	second-order tensor $= (\partial \mathbf{e}'_i/\partial \dot{\mathbf{r}})\,|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle}$
$\mathbf{b}$	unit vector normal to the tangent of the mean reinforcement path
$\hat{\mathbf{c}}^{(i)}$	third-order tensor $= (\partial^2 \mathbf{e}'_i/\partial \dot{\mathbf{r}}\,\partial \dot{\mathbf{r}})\,|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle}$
$\hat{\mathbf{C}}$	fourth-order stiffness tensor
C_{ijkl}	components of the stiffness tensor
$\mathbf{C}$	column-vector which elements are components of the stiffness tensor
$\mathbf{e}_i$	global orthonormal basis
$\mathbf{e}'_i$	local orthonormal basis
$\mathbf{e}'_{ij}$	directional cosines $= \mathbf{e}'_i \cdot \mathbf{e}_j$
$\mathbf{e}'$	set of local basis vectors $= \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$
$\hat{\mathbf{I}}$	second-order identity tensor
$\hat{\mathbf{K}}_{\mathbf{XX}}$	covariance tensor $= \langle \mathring{\mathbf{X}} \otimes \mathring{\mathbf{X}} \rangle$
$\langle L^{(k)} \rangle$	length of the mean reinforcement path
$\mathbf{n}$	unit vector normal to tangent of the mean reinforcement path
$\mathbf{r}$	position vector of the reinforcement path
$\hat{\mathbf{S}}$	fourth-order compliance tensor
S_{ijkl}	components of the compliance tensor
$\mathbf{S}$	column-vector which elements are components of the compliance tensor
S^0_{ij}	compliances of a unidirectional composite with straight fibers referred to principal axes of the material
$\mathbf{t}$	unit tangent vector of the mean reinforcement path
V	total volume of the composite
V_m	total matrix volume in the composite
V_f	fiber volume fraction
x_i	global coordinates
$\Omega^{(k)}$	set of stochastically identical reinforcement paths
ε_{ijk}	Levi–Civita tensor density
$\mu^{(k)}$	relative volumetric fraction of the kth subcomposite
δ	Dirac delta function
δ_{ij}	Kronecker delta
$\hat{\boldsymbol{\sigma}}$	stress tensor
σ_{ij}	components of the stress tensor
$\hat{\boldsymbol{\varepsilon}}$	strain tensor

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ε_{ij} components of the strain tensor
φ_k set of the coordinate functions in canonical expansion of the stochastic reinforcement path
λ an arbitrary vector not parallel to the tangent line of the reinforcement path
θ local deflection angle
σ_θ standard deviation of the deflection angle of the reinforcement path
ξ running parameter of the reinforcement path

Special notations:
$\langle \mathbf{X} \rangle$ mean of random function $\mathbf{X}$
$\mathring{\mathbf{X}}$ centered random function $= \mathbf{X} - \langle \mathbf{X} \rangle$
$\dot{\mathbf{X}}, \ddot{\mathbf{X}}$ first and second derivatives with respect to the parameter ξ
$\mathbf{X} \cdot \mathbf{Y}$ scalar product
$\hat{\mathbf{X}} : \hat{\mathbf{Y}}$ doubly contracted product
$\hat{\mathbf{X}} \times \hat{\mathbf{Y}}$ vector product
$\hat{\mathbf{X}} \otimes \hat{\mathbf{Y}}$ outer tensor product
$\hat{\mathbf{X}}^T$ transposed tensor
$|\mathbf{X}|$ norma $= \sqrt{\mathbf{X} \cdot \mathbf{X}}$

Notes:
Bold faced letter denotes a vector.
Bold faced letter with an overcap symbol "^" denotes a tensor.

1. INTRODUCTION

Fiber curvatures and/or layer waviness are inevitable technological factors in composites manufacturing, e.g. thermal processing, filament winding, braiding, weaving, stitching, etc. Both curved fibers and wavy layers can significantly affect the composite material properties and, consequently, the structural performance.

For braided composites, variations in crimp angle, braid angle, and local volume fraction are intrinsic. While the above parameters are not mutually independent, the braid angle is clearly the parameter which is most sensitive to the processing errors. Minor local deviations in braid angle can occur as a result of variations in rotational/translational speeds while braiding, handling of the material prior to impregnation, and resin infiltration. Waviness and other imperfections can be relatively high. Usually, it is difficult even to categorize the forms of imperfections exhibited by textile composites, let along to precisely measure them all (Cox, 1995). Hence the development of probabilistic approaches aimed at predicting thermomechanical response and sensitivity of composites having stochastic reinforcement imperfections is an important issue from both scientific and engineering standpoints. In a deterministic analysis, the effect of the reinforcement waviness and other structural imperfections on the performance of composite materials have been studied extensively using both experimental (Kuo *et al.*, 1988; Clyburn, 1993; Chen *et al.*, 1996) and theoretical (Pastore and Gowayed, 1994; Liu and Xu, 1995; Gowayed *et al.*, 1996; Sun *et al.*, 1995; Xu *et al.*, 1995) methods. Probabilistic aspects of the problem are far less developed. There are several optional approaches to study the effect of stochastic structural parameters of the reinforcement on the composite performance:

- Monte Carlo technique.
- Direct formal averaging of elastic constants of a homogeneous anisotropic material.
- Application of the theory of stochastic functions for analyzing the response of the material which includes reinforcement imperfections.

Monte Carlo approach is currently most popular; this is a rather universal tool, but computationally very expensive. Monte Carlo technique was used by Pastore (1993) to analyze effects of local variations in the reinforcement on the global stiffness variation of textile composite. Another approach based on direct formal averaging (Chou and Takanashi, 1987; Cox, 1995) assumes that the misalignment angle can be characterized by a normal one-dimensional differential probability function (DPF) with the prescribed mean value and standard deviation. Formal averaging of the corresponding deterministic equations with respect to the misalignment angle provides the mean values of the elastic characteristics. In fact, this approach can be viewed as the extension of the methods developed earlier for the analysis of composites with misaligned short straight fibers (Cox, 1952; Cook, 1968). For the problems where values of random functions at various points of space do not need to be interrelated (like in the case of misaligned short straight-fiber composites), the one-dimensional DPF is an adequate characteristic of the stochastic reinforcement imperfections. However, the approach is not able to characterize spatial variations of the imperfections. Apparently, a unidirectional composite with continuous wavy fiber reinforcement represents the simplest example of the structure which cannot be adequately characterized using the one-dimensional DPF.

The theory of random functions was applied by Bolotin (1966) to analyze laminated medium reinforced with slightly curved elastic layers. It was assumed that the initial waviness of the reinforced layers form a stationary stochastic field. The method of canonical expansions has been used to derive the statistical characteristics of the stresses, strains and displacements of the stochastic medium. The theory was used to explain experimentally observed significant reduction of the elastic moduli of layered glass-reinforced plastics vs theoretical predictions based on the assumption of "perfect" reinforcement.

In the present study, a novel stochastic theory of composite materials with continuous-fiber reinforcements is developed. The spatial reinforcement path is treated as a random vector function characterized in 3-D space by its mean value and covariance matrix. By introducing local stochastic basis at any point along the reinforcement path and further applying the series expansion technique, the mean value and covariance of elastic characteristics of the composite material are expressed in terms of the mean value and covariance of the reinforcement path. The theory assumes that the fluctuations of the reinforcement path are rather small compared to the characteristic scale of its mean value. The final result is obtained in terms of the second-order approximation of the mean and the first-order approximation of the covariance derived for the elastic characteristics. Specific solutions are then obtained for a unidirectional composite with wavy fibers and helical wound composite. It is shown that the earlier results of Bolotin (1966) and Cox (1995) for unidirectional composite with random waviness follow from the developed theory.

The theory is aimed at developing efficient analytical and computational tools for stochastic problems of composite materials. The output of the theory, namely, the calculated mean values and standard deviations of stiffness and compliance tensors of the material provides the necessary input data for the reliability analysis of the composite structural elements (Yushanov and Bogdanovich, 1998). It has to be emphasized that the issue of obtaining complete and reliable input data needed for any kind of stochastic analysis of composites still remains the major obstacle towards practical applications.

2. THEORETICAL CHARACTERIZATION OF A STOCHASTIC REINFORCEMENT PATH

2.1. *Covariance function of the reinforcement path*

Consider some arbitrary curve P_0P_1 in 3-D space which will be called a "reinforcement path", Fig. 1. A set of orthogonal axes identified by symbols x_1, x_2, x_3 refers to the global right-handed coordinate system. A triad of unit vectors $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ is an orthonormal basis of the global coordinate system, hence

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{1}$$

where symbol "·" denotes scalar (dot) product and δ_{ij} is the Kronecker delta. The reinforcement path P_0P_1 is specified by a position vector in a parametric form

$$\mathbf{r}(\xi) = x_i(\xi)\mathbf{e}_i \tag{2}$$

where ξ is the running parameter. In particular, ξ can represent the arc length of the mean reinforcement path. As ξ varies from ξ_0 to ξ_1, the end point of the position vector $\mathbf{r}(\xi)$ traces out the reinforcement path from point P_0 to point P_1. Summation convention is used in eqn (2). Namely, every letter index appearing twice in one term is regarded as summation index.

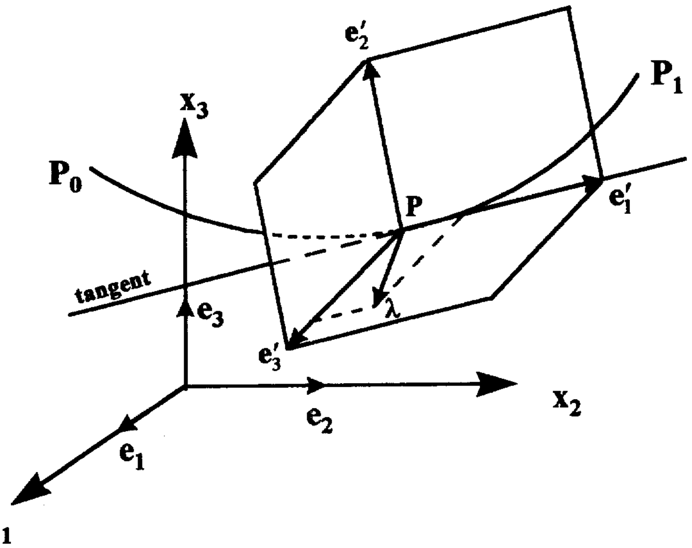

Fig. 1. Global coordinate system, $\{x_1, x_2, x_3\}$, and local basis, $\{\mathbf{e}_i'\}$, related to the stochastic reinforcement path P_0P_1.

An arbitrary reinforcement path can be represented by a canonical expansion (Pugachev, 1965) of a random vector function which defines the curved reinforcement path as follows:

$$\mathbf{r}(\xi) = \Phi_0(\xi) + \sum_{k=1}^{\infty} \mathbf{V}_k \varphi_k(\xi). \tag{3}$$

Here $\Phi_0(\xi)$ is some prescribed deterministic vector-function, $\{\varphi_k(\xi)\}$ is a set of deterministic coordinate basis functions, and $\{\mathbf{V}_k\}$ is a set of orthogonal zero-mean random values. Expansion (3) decomposes $\mathbf{r}(\xi)$ into a purely deterministic component and a purely random component. The random component of $\mathbf{r}(\xi)$ is completely identified by the distribution densities of the coefficients $\mathbf{V}_k$. However, it is more convenient to use numerical characteristics (moments) of $\mathbf{V}_k$ instead of the distribution densities. The fundamental properties of a random function are characterized by its first- and second-order moments, which are customary in most of the practical applications. The moments are defined as follows:

$$\langle \mathbf{r}(\xi) \rangle = \Phi_0(\xi)$$

$$\hat{\mathbf{K}}_{rr}(\xi, \varsigma) = \langle \mathring{\mathbf{r}}(\xi) \otimes \mathring{\mathbf{r}}(\xi+\varsigma) \rangle = \sum_{k=1}^{\infty} \langle \mathbf{V}_k \otimes \mathbf{V}_k \rangle \varphi_k(\xi) \varphi_k(\xi+\varsigma) \tag{4}$$

where $\mathring{\mathbf{r}}(\xi) = \mathbf{r}(\xi) - \langle \mathbf{r}(\xi) \rangle$, and symbol "$\otimes$" denotes outer tensor product. Function $\langle \mathbf{r}(\xi) \rangle$ is called the mean (or mathematical expectation) of $\mathbf{r}(\xi)$, and the second centered moment $\hat{\mathbf{K}}_{rr}(\xi, \varsigma)$ is the covariance function. Matrix form of the covariance function of the stochastic vector function is as follows:

$$\hat{\mathbf{K}}_{rr}(\xi, \varsigma) = \begin{bmatrix} K_{x_1x_1}(\xi, \varsigma) & K_{x_1x_2}(\xi, \varsigma) & K_{x_1x_3}(\xi, \varsigma) \\ & K_{x_2x_2}(\xi, \varsigma) & K_{x_2x_3}(\xi, \varsigma) \\ \text{symm} & & K_{x_3x_3}(\xi, \varsigma) \end{bmatrix} \tag{5}$$

where $K_{x_ix_j}(\xi, \varsigma) = \langle \mathring{x}_i(\xi) \mathring{x}_j(\xi+\varsigma) \rangle$. Further, covariance of the first derivative of the function $\mathbf{r}(\xi)$ is defined as

$$\hat{\mathbf{K}}_{\dot{r}\dot{r}}(\xi,\varsigma) = \frac{\partial^2 \hat{\mathbf{K}}_{rr}(\xi,\varsigma)}{\partial\xi\,\partial\varsigma} \tag{6}$$

where $\dot{\mathbf{r}}(\xi) = \mathrm{d}\dot{\mathbf{r}}(\xi)/\mathrm{d}\xi$. From the geometrical standpoint, $\dot{\mathbf{r}}(\xi)$ is a local tangent to the reinforcement path.

2.2. *Local stochastic basis*

A triad of unit vectors $\{\mathbf{e}'_i\}$ at any point along the stochastic reinforcement path form a stochastic local orthonormal basis. If unit vector $\mathbf{e}'_1$ is selected as the tangent vector of the reinforcement path, then orientation of the other two unit vectors, $\mathbf{e}'_2$ and $\mathbf{e}'_3$, can be chosen voluntarily. For example, the unit vector $\mathbf{e}'_2$ can be directed in the plane which contains vectors $\boldsymbol{\lambda}$ and $\mathbf{e}'_1$, where $\boldsymbol{\lambda}$ is an arbitrary unit vector ($\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = 1$); the only condition is that $\boldsymbol{\lambda}\times\mathbf{e}'_1 \neq 0$. This defines uniquely a local stochastic orthonormal basis:

$$\mathbf{e}'_1 = \frac{\dot{\mathbf{r}}(\xi)}{|\dot{\mathbf{r}}(\xi)|}, \quad \mathbf{e}'_2 = \frac{\boldsymbol{\lambda}\times\dot{\mathbf{r}}(\xi)}{|\boldsymbol{\lambda}\times\dot{\mathbf{r}}(\xi)|}, \quad \mathbf{e}'_3 = \mathbf{e}'_1\times\mathbf{e}'_2. \tag{7}$$

The singular points where $|\dot{\mathbf{r}}(\xi)| = 0$ are excluded from the consideration. The stochastic coordinates of the local basis are then obtained as the stochastic directional cosines, e'_{ij}, between the local and global axes. For example, $e'_{12} = \mathbf{e}'_1\cdot\mathbf{e}_2 = \dot{x}_2(\xi)/\sqrt{\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2}$ is the directional cosine between the local unit tangent vector $\mathbf{e}'_1$ and global axes $\mathbf{x}_2$, etc.

Equations (7) uniquely define directional cosines of an arbitrary reinforcement path specified by the position vector eqn (2). The local basis is thus, tied to each point of the reinforcement path, and since fibers are stochastically curved, the local basis has stochastic orientation.

2.3. *Expansion of the local stochastic basis vectors*

To obtain the mean and covariance of the local basis related to the stochastic reinforcement path, an expansion procedure of the local basis about the mean local tangent line is applied. In the developed procedure, the vectors $\mathbf{e}'_2$ and $\mathbf{e}'_3$ are expanded into Taylor's series about all of their arguments. Expansion of the local basis vectors eqn (7) into Taylor's series about the point $\dot{\mathbf{r}} = \langle\dot{\mathbf{r}}\rangle$ yields:

$$\mathbf{e}'_i = \mathbf{e}'_i|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle} + \left.\frac{\partial\mathbf{e}'_i}{\partial\dot{\mathbf{r}}}\right|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle}\cdot(\dot{\mathbf{r}}-\langle\dot{\mathbf{r}}\rangle) + \frac{1}{2}\left.\frac{\partial^2\mathbf{e}'_i}{\partial\dot{\mathbf{r}}\,\partial\dot{\mathbf{r}}}\right|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle} : (\dot{\mathbf{r}}-\langle\dot{\mathbf{r}}\rangle)\otimes(\dot{\mathbf{r}}-\langle\dot{\mathbf{r}}\rangle) + \cdots. \tag{8}$$

Here, notation ":" for doubly contracted product is used. Recall that the doubly contracted product reduces the order of the resulting product by two.

The first three terms of expansion eqn (8) play a significant role in the analysis. The first term represents directional cosines of the mean reinforcement path. The second term defines covariance properties of the local basis. The third term provides correction to the mean value of the directional cosines due to assumed random fluctuations around the mean reinforcement path. Evaluation of the coefficients in eqn (8) yields

$$\mathbf{e}'_i = \mathbf{a}^{(i)} + \hat{\mathbf{b}}^{(i)}\cdot\mathring{\dot{\mathbf{r}}} + \tfrac{1}{2}\hat{\mathbf{c}}^{(i)} : \mathring{\dot{\mathbf{r}}}\otimes\mathring{\dot{\mathbf{r}}} \tag{9}$$

where $\mathring{\dot{\mathbf{r}}}(\xi) = \dot{\mathbf{r}}(\xi) - \langle\dot{\mathbf{r}}(\xi)\rangle$ is the centered stochastic tangent vector to the reinforcement path. Coefficients $\mathbf{a}^{(i)} = \mathbf{e}'_i(\dot{\mathbf{r}})\,|_{\dot{\mathbf{r}}=\langle\dot{\mathbf{r}}\rangle}$ are respective vectors of the local basis evaluated at $\dot{\mathbf{r}} = \langle\dot{\mathbf{r}}\rangle$:

$$\mathbf{a}^{(1)} = \frac{\langle \dot{\mathbf{r}} \rangle}{|\langle \dot{\mathbf{r}} \rangle|}$$

$$\mathbf{a}^{(2)} = \frac{\boldsymbol{\lambda} \times \langle \dot{\mathbf{r}} \rangle}{|\boldsymbol{\lambda} \times \langle \dot{\mathbf{r}} \rangle|}$$

$$\mathbf{a}^{(3)} = \mathbf{a}^{(1)} \times \mathbf{a}^{(2)}. \tag{10}$$

Obviously, $\mathbf{a}^{(i)}$ is an orthonormal triad related to the mean reinforcement path, and $\mathbf{a}^{(1)}$ is the tangent unit vector of the mean reinforcement path. Coefficients $\hat{\mathbf{b}}^{(i)} = (\partial \mathbf{e}'_i(\dot{\mathbf{r}})/\partial \dot{\mathbf{r}})\,|_{\dot{\mathbf{r}}=\langle \dot{\mathbf{r}} \rangle}$ are the second-order tensors expressed as

$$\hat{\mathbf{b}}^{(1)} = \frac{\hat{\mathbf{I}} - \mathbf{a}^{(1)} \otimes \mathbf{a}^{(1)}}{\langle \dot{\mathbf{r}} \rangle}$$

$$\hat{\mathbf{b}}^{(2)} = \frac{\hat{\mathbf{p}} + \mathbf{a}^{(2)} \otimes \mathbf{s}}{|\boldsymbol{\lambda} \times \langle \dot{\mathbf{r}} \rangle|}$$

$$\hat{\mathbf{b}}^{(3)} = \mathbf{a}^{(1)} \times \hat{\mathbf{b}}^{(2)} - \mathbf{a}^{(2)} \times \hat{\mathbf{b}}^{(1)} \tag{11}$$

where $\mathbf{s} = \boldsymbol{\lambda} \times \mathbf{a}^{(2)}$, $\hat{\mathbf{p}} = \boldsymbol{\lambda} \times \hat{\mathbf{I}}$ is the second-order tensor, and $\hat{\mathbf{I}}$ is the second-order identity tensor. Finally, the coefficients $\hat{\mathbf{c}}^{(i)} = (\partial^2 \mathbf{e}'_i(\dot{\mathbf{r}})/\partial \dot{\mathbf{r}}\, \partial \dot{\mathbf{r}})\,|_{\dot{\mathbf{r}}=\langle \dot{\mathbf{r}} \rangle}$ are the third-order tensors defined by the following relations

$$\hat{\mathbf{c}}^{(1)} = -\frac{\mathbf{a}^{(1)} \otimes \hat{\mathbf{I}} + \hat{\mathbf{I}} \otimes \mathbf{a}^{(1)} + (\mathbf{a}^{(1)} \otimes \hat{\mathbf{I}})^T + 3\mathbf{a}^{(1)} \otimes \mathbf{a}^{(1)} \otimes \mathbf{a}^{(1)}}{|\langle \dot{\mathbf{r}} \rangle|^2}$$

$$\hat{\mathbf{c}}^{(2)} = \frac{\hat{\mathbf{p}} \otimes \mathbf{s} + (\mathbf{s} \otimes \hat{\mathbf{p}})^T + \mathbf{a}^{(2)} \otimes (3\mathbf{s} \otimes \mathbf{s} + \boldsymbol{\lambda} \otimes \boldsymbol{\lambda} - \hat{\mathbf{I}})}{|\boldsymbol{\lambda} \times \langle \dot{\mathbf{r}} \rangle|^2}$$

$$\hat{\mathbf{c}}^{(3)} = \mathbf{a}^{(1)} \times \hat{\mathbf{c}}^{(2)} - \mathbf{a}^{(2)} \times \hat{\mathbf{c}}^{(1)} + (\hat{\mathbf{b}}^{(1)T} \times \hat{\mathbf{b}}^{(2)})^T - (\hat{\mathbf{b}}^{(2)T} \times \hat{\mathbf{b}}^{(1)})^T. \tag{12}$$

It should be emphasized that operation of the transposition applied to the third-order tensor transposes the first two indices in a tensor. The following identities have been used in the derivation of eqns (11) and (12):

$$\frac{\partial(\boldsymbol{\lambda} \times \dot{\mathbf{r}})}{\partial \dot{\mathbf{r}}} = \boldsymbol{\lambda} \times \hat{\mathbf{I}}, \quad \frac{\partial}{\partial \dot{\mathbf{r}}}\left(\frac{1}{|\dot{\mathbf{r}}|^\alpha}\right) = -\alpha \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^{\alpha+2}}, \quad \frac{\partial}{\partial \dot{\mathbf{r}}}\left(\frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^\alpha}\right) = \alpha \frac{\boldsymbol{\lambda} \times \mathbf{e}'_2}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+1}}, \tag{13}$$

where $\alpha > 0$. Details of the derivations of eqns (11) and (12) are given in Appendix A.

2.4. *Mean and covariance of the local basis vectors*

Averaging of eqn (9) provides the second-order approximation of the mean basis

$$\langle \mathbf{e}'_i \rangle = \mathbf{a}^{(i)} + \tfrac{1}{2} \hat{\mathbf{c}}^{(i)} : \langle \mathring{\dot{\mathbf{r}}} \otimes \mathring{\dot{\mathbf{r}}} \rangle. \tag{14}$$

The term $\langle \mathring{\dot{\mathbf{r}}} \otimes \mathring{\dot{\mathbf{r}}} \rangle$ is the covariance of the tangent vector along the reinforcement path $\dot{\mathbf{r}}(\xi)$. This is defined, according to eqn (6), as $\hat{\mathbf{K}}_{\dot{r}\dot{r}}(\xi, \xi) = \langle \mathring{\dot{\mathbf{r}}}(\xi) \otimes \mathring{\dot{\mathbf{r}}}(\xi) \rangle = \ddot{\hat{\mathbf{K}}}_{rr}(\xi, \xi)$. Hence, the mean values of the local basis vectors are expressed in the form

$$\langle \mathbf{e}'_i \rangle = \mathbf{a}^{(i)} + \tfrac{1}{2} \hat{\mathbf{c}}^{(i)} : \ddot{\hat{\mathbf{K}}}_{rr}(\xi, \xi). \tag{15}$$

The centered local basis is of the form

$$\mathring{\mathbf{e}}'_i = \mathbf{e}'_i - \langle \mathbf{e}'_i \rangle = \hat{\mathbf{b}}^{(i)} \cdot \mathring{\mathbf{r}}. \tag{16}$$

Thus, the first-order approximation of the covariances of the local basis vectors is obtained from eqn (16) as following

$$\langle \mathring{\mathbf{e}}'_i \otimes \mathring{\mathbf{e}}'_j \rangle = \hat{\mathbf{b}}^{(i)} \cdot \ddot{\mathbf{K}}_{rr}(\xi, \xi) \cdot (\hat{\mathbf{b}}^{(j)})^T. \tag{17}$$

For the further convenience, the nine-component vector of directional cosines $\mathbf{e}' = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is introduced. Then, the covariance $\hat{\mathbf{K}}_{\mathbf{e}'\mathbf{e}'}$ of the local basis vectors can be written in the matrix form as

$$\hat{\mathbf{K}}_{\mathbf{e}'\mathbf{e}'} = \langle \mathring{\mathbf{e}}' \otimes \mathring{\mathbf{e}}' \rangle = \begin{bmatrix} \langle \mathring{\mathbf{e}}'_1 \otimes \mathring{\mathbf{e}}'_1 \rangle & \langle \mathring{\mathbf{e}}'_1 \otimes \mathring{\mathbf{e}}'_2 \rangle & \langle \mathring{\mathbf{e}}'_1 \otimes \mathring{\mathbf{e}}'_3 \rangle \\ & \langle \mathring{\mathbf{e}}'_2 \otimes \mathring{\mathbf{e}}'_2 \rangle & \langle \mathring{\mathbf{e}}'_2 \otimes \mathring{\mathbf{e}}'_3 \rangle \\ \text{symm} & & \langle \mathring{\mathbf{e}}'_3 \otimes \mathring{\mathbf{e}}'_3 \rangle \end{bmatrix}. \tag{18}$$

Each element of eqn (18) is 3×3 submatrix defined by eqn (17).

3. EVALUATION OF THE MEAN AND COVARIANCE OF THE STIFFNESS AND COMPLIANCE TENSORS

Stiffness, $\hat{\mathbf{C}}$, and compliance, $\hat{\mathbf{S}}$, are fourth-order tensors defined by constitutive relations of an anisotropic material

$$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{C}} : \hat{\boldsymbol{\varepsilon}}, \quad \hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{S}} : \hat{\boldsymbol{\sigma}} \tag{19}$$

where $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\varepsilon}}$ are second-order symmetric stress and strain tensors, respectively. In the component form, constitutive relations eqn (19) take the form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{lk}, \quad \varepsilon_{ij} = S_{ijkl}\sigma_{lk}(i,j,k,l = 1,2,3). \tag{20}$$

Stiffness and compliance tensors referred to the "global" coordinate system are related to the stiffness, $\hat{\mathbf{C}}'$, and compliance, $\hat{\mathbf{S}}'$, tensors referred to the "local" coordinate system through the following tensor transformation law

$$C_{ijkl} = C'_{mnop} e'_{mi} e'_{nj} e'_{ok} e'_{pl}, \quad S_{ijkl} = S'_{mnop} e'_{mi} e'_{nj} e'_{ok} e'_{pl}. \tag{21}$$

It is suitable for the forthcoming derivations to represent stiffness matrix related to global coordinate system in a column-vector form $\mathbf{C} = \{C_{1111}, C_{1122}, \ldots\}^T$. For a generally anisotropic material, vector $\mathbf{C}$ has 21 non-zero, independent components. Analogously, a 21-component stiffness column-vector $\mathbf{C}' = \{C'_{1111}, C'_{1122}, \ldots\}^T$ related to the local coordinate system is introduced. According to eqn (21), each element of the column-vector $\mathbf{C}$ is the function of the components of the vector $e' = \{e'_1, e'_2, e'_3\} = \{e'_{11}, e'_{12}, e'_{13}, e'_{21}, e'_{22}, e'_{23}, e'_{31}, e'_{32}, e'_{33},\}$ and the components of the stiffness vector $\mathbf{C}'$. The functional relation is symbolically written as:

$$\mathbf{C} = \mathbf{C}(\mathbf{C}', \mathbf{e}'). \tag{22}$$

Here and henceforth we assume that the local stiffness, $\mathbf{C}'$, is deterministic. However, since any local basis is stochastic, the global stiffness vector $\mathbf{C}$ is a stochastic one. Defining probabilistic characteristics of its components is our next objective.

To evaluate mean and covariance matrix of a stochastic column-vector $\mathbf{C}$, the power-series expansion about the mean local basis is applied:

$$\mathbf{C} = \mathbf{C}|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} + \left.\frac{\partial \mathbf{C}}{\partial \mathbf{e}'}\right|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} \cdot (\mathbf{e}' - \langle\mathbf{e}'\rangle) + \frac{1}{2}\left.\frac{\partial^2 \mathbf{C}}{\partial \mathbf{e}'\,\partial \mathbf{e}'}\right|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} : (\mathbf{e}' - \langle\mathbf{e}'\rangle) \otimes (\mathbf{e}' - \langle\mathbf{e}'\rangle) + \cdots. \tag{23}$$

By introducing notations

$$\hat{\mathbf{d}}^{(\mathbf{C})} = \left.\frac{\partial \mathbf{C}}{\partial \mathbf{e}'}\right|_{\mathbf{e}'=\langle\mathbf{e}'\rangle}, \quad \hat{\mathbf{f}}^{(\mathbf{C})} = \left.\frac{\partial^2 \mathbf{C}}{\partial \mathbf{e}'\,\partial \mathbf{e}'}\right|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} \tag{24}$$

and disregarding the small terms of order $|\mathbf{e}' - \langle\mathbf{e}'\rangle|^3$ and higher, eqn (23) reduces to

$$\mathbf{C} = \mathbf{C}|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} + \hat{\mathbf{d}}^{(\mathbf{C})} \cdot (\mathbf{e}' - \langle\mathbf{e}'\rangle) + \tfrac{1}{2}\hat{\mathbf{f}}^{(\mathbf{C})} : (\mathbf{e}' - \langle\mathbf{e}'\rangle) \otimes (\mathbf{e}' - \langle\mathbf{e}'\rangle). \tag{25}$$

Then, using eqn (25), the mean and covariance of $\mathbf{C}$ are obtained as

$$\langle \mathbf{C} \rangle = \mathbf{C}|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} + \tfrac{1}{2}\hat{\mathbf{f}}^{(\mathbf{C})} : \hat{\mathbf{K}}_{\mathbf{e}'\mathbf{e}'} \tag{26}$$

$$\langle \mathring{\mathbf{C}} \otimes \mathring{\mathbf{C}} \rangle = \hat{\mathbf{d}}^{(\mathbf{C})} \cdot \hat{\mathbf{K}}_{\mathbf{e}'\mathbf{e}'} \cdot (\hat{\mathbf{d}}^{(\mathbf{C})})^T. \tag{27}$$

According to eqn (24), $\hat{\mathbf{d}}^{(\mathbf{C})}$ is a 21×9 array and $\hat{\mathbf{f}}^{(\mathbf{C})}$ is a $21 \times 9 \times 9$ array. The components of these arrays are calculated as

$$d_{ij}^{(\mathbf{C})} = \left.\frac{\partial C_i}{\partial e_j'}\right|_{\mathbf{e}'=\langle\mathbf{e}'\rangle}, \quad f_{ijk}^{(\mathbf{C})} = \left.\frac{\partial^2 C_i}{\partial e_j'\,\partial e_k'}\right|_{\mathbf{e}'=\langle\mathbf{e}'\rangle}. \tag{28}$$

Similarly, the global compliance column-vector is introduced as $\mathbf{S} = \{S_{1111}, S_{1122}, \ldots\}^T$. Its corresponding mean and covariance matrix are

$$\langle \mathbf{S} \rangle = \mathbf{S}|_{\mathbf{e}'=\langle\mathbf{e}'\rangle} + \tfrac{1}{2}\hat{\mathbf{f}}^{(\mathbf{S})} : \hat{\mathbf{K}}_{\mathbf{e}'\mathbf{e}'} \tag{29}$$

$$\langle \mathring{\mathbf{S}} \otimes \mathring{\mathbf{S}} \rangle = \hat{\mathbf{d}}^{(\mathbf{S})} \cdot \hat{\mathbf{K}}_{\mathbf{e}'\mathbf{e}'} \cdot (\hat{\mathbf{d}}^{(\mathbf{S})})^T \tag{30}$$

where $\hat{\mathbf{f}}^{(\mathbf{S})} = (\partial^2 \mathbf{S}/\partial \mathbf{e}'\,\partial \mathbf{e}')\,|_{\mathbf{e}'=\langle\mathbf{e}'\rangle}$ and $\hat{\mathbf{d}}^{(\mathbf{S})} = (\partial \mathbf{S}/\partial \mathbf{e}')\,|_{\mathbf{e}'=\langle\mathbf{e}'\rangle}$.

The mean $\langle \hat{\mathbf{C}} \rangle$ and covariance matrix $\langle \mathring{\hat{\mathbf{C}}} \otimes \mathring{\hat{\mathbf{C}}} \rangle$ of the global stiffness tensor can then be obtained by rearranging components of eqns (26) and (27), respectively, back to the tensor form. Similarly, mean and covariance matrix of the global compliance tensor, $\langle \hat{\mathbf{S}} \rangle$ and $\langle \mathring{\hat{\mathbf{S}}} \otimes \mathring{\hat{\mathbf{S}}} \rangle$, can be obtained by rearranging components of the vector eqn (29) and matrix eqn (30), respectively.

Equations (26), (27) and (29), (30) provide the mean values and covariance functions of the stiffnesses and compliances at an arbitrary point along the reinforcement path. The averaging procedure over all of the reinforcement paths has to be next applied to evaluate the stochastic elastic characteristics (mean and covariance) of the composite media.

4. GLOBAL STIFFNESS/COMPLIANCE AVERAGING

Consider some volume V of reinforced composite with arbitrary number of reinforcement paths. Suppose that each kind of the reinforcement paths can be represented as a countable set, $\Omega^{(k)}$, all elements of which are stochastically identical. Superscript "k" will be used to denote a particular set of the reinforcement paths. Each set is specified by the stochastic position vector $\mathbf{r}^{(k)}(\xi)$. The reinforcement paths $\mathbf{r}^{(k)}(\xi) \in \Omega^{(k)}$ are defined as stochastically identical if all of them are characterized by the same mean position vector $\langle \mathbf{r}^{(k)}(\xi) \rangle$ and covariance matrix $\mathbf{K}_{\mathbf{r}^{(k)}\mathbf{r}^{(k)}}(\xi, \xi) = \langle \mathring{\mathbf{r}}^{(k)}(\xi) \otimes \mathring{\mathbf{r}}^{(k)}(\xi) \rangle$.

Denote by $V_f^{(k)}$ the fiber volume fraction of all reinforcement paths belonging to the set $\Omega^{(k)}$. The total fiber volume fraction of all kind reinforcements in the composite is thus $V_f = \Sigma_k V_f^{(k)}$. A new object of consideration, to be called a "subcomposite" is now introduced. The kth subcomposite is defined as a set of stochastically identical reinforcement paths surrounded by some amount of the matrix material. The total amount of matrix distributed among the subcomposites is defined according to the following rule: matrix volume $V_m^{(k)}$ attached to the kth subcomposite is $V_m^{(k)} = V_m(V_f^{(k)}/V_f)$, where V_m is total matrix volume in the composite. The consequence of this assumption is that the fiber volume fraction in each subcomposite is the same as the fiber volume fraction in the overall composite. The deterministic analog of this assumption has been introduced and analyzed by Kregers and Melbardis (1978), Kregers (1979, 1982).

Volumetric averaging of the stiffness tensor over all sets of stochastically identical reinforcement paths gives the following expression of the total stiffness of the composite:

$$\hat{\mathbf{C}}^{(\text{aver})} = \frac{1}{V}\sum_k \int_{V^{(k)}} \hat{\mathbf{C}}^{(k)}\,\mathrm{d}V^{(k)} \tag{31}$$

where $\hat{\mathbf{C}}^{(k)}$ is the stiffness tensor of the kth subcomposite and $V^{(k)}$ denotes the volume of the kth subcomposite. Volumetric integration in eqn (31) can be reduced to the line integration along the mean reinforcement path:

$$\hat{\mathbf{C}}^{(\text{aver})} = \sum_k \mu^{(k)} \frac{1}{\langle L^{(k)}\rangle} \int_{\langle \mathbf{r}^{(k)}\rangle} \hat{\mathbf{C}}^{(k)}\,\mathrm{d}\langle L^{(k)}\rangle \tag{32}$$

where $\mu^{(k)} = V^{(k)}/\Sigma_k V^{(k)} = V^{(k)}/V$ is the volume fraction of the kth subcomposite, $\mathrm{d}\langle L^{(k)}\rangle = \sqrt{\langle \dot{\mathbf{r}}^{(k)}\rangle \cdot \langle \dot{\mathbf{r}}^{(k)}\rangle}\,\mathrm{d}\xi$ is the element of the arc length of the mean reinforcement path corresponding to the kth stochastic reinforcement set, and $\langle L^{(k)}\rangle = \int_{\xi_0}^{\xi_1} \sqrt{\langle \dot{\mathbf{r}}^{(k)}\rangle \cdot \langle \dot{\mathbf{r}}^{(k)}\rangle}\,\mathrm{d}\xi$ is the length of the kth mean reinforcement path. If parameter ξ is the arc length of the mean reinforcement path, then $|\langle \dot{\mathbf{r}}^{(k)}\rangle| \equiv 1$ and $\mathrm{d}\langle L^{(k)}\rangle = \mathrm{d}\xi$.

According to eqn (32), the total stiffness is obtained as the superposition of the stiffnesses corresponding to all individual sets $\Omega^{(k)}$ of the reinforcement paths. Hence, no interaction (in the stochastic sense) between the reinforcement sets is taken into account in this theory. The mean and covariance of the averaged stiffness tensor of the composite are thus evaluated as

$$\begin{aligned} \langle \hat{\mathbf{C}}^{(\text{aver})}\rangle &= \sum_k \mu^{(k)} \frac{1}{\langle L^{(k)}\rangle} \int_{\langle \mathbf{r}^{(k)}\rangle} \langle \hat{\mathbf{C}}^{(k)}\rangle\,\mathrm{d}\langle L^{(k)}\rangle \\ \langle \mathring{\hat{\mathbf{C}}}^{(\text{aver})} \otimes \mathring{\hat{\mathbf{C}}}^{(\text{aver})}\rangle &= \sum_k \mu^{(k)} \frac{1}{\langle L^{(k)}\rangle} \int_{\langle \mathbf{r}^{(k)}\rangle} \langle \mathring{\hat{\mathbf{C}}}^{(k)} \otimes \mathring{\hat{\mathbf{C}}}^{(k)}\rangle\,\mathrm{d}\langle L^{(k)}\rangle \end{aligned} \tag{33}$$

where $\langle \hat{\mathbf{C}}^{(k)}\rangle$ and $\langle \mathring{\hat{\mathbf{C}}}^{(k)} \otimes \mathring{\hat{\mathbf{C}}}^{(k)}\rangle$ are expressed by eqns (26) and (27) for each specific reinforcement path $\mathbf{r}^{(k)}(\xi)$.

Using similar considerations, the compliance averaging approach results in

$$\hat{\mathbf{S}}^{(\text{aver})} = \sum_k \mu^{(k)} \frac{1}{\langle L^{(k)}\rangle} \int_{\langle \mathbf{r}^{(k)}\rangle} \hat{\mathbf{S}}^{(k)}\,\mathrm{d}\langle L^{(k)}\rangle \tag{34}$$

where $\hat{\mathbf{S}}^{(k)}$ is the local compliance of the kth set subcomposite. Then the mean and covariance of the averaged compliance tensor of the composite are

$$\langle \hat{\mathbf{S}}^{(\text{aver})} \rangle = \sum_k \mu^{(k)} \frac{1}{\langle L^{(k)} \rangle} \int_{\langle \mathbf{r}^{(k)} \rangle} \langle \hat{\mathbf{S}}^{(k)} \rangle \, \mathrm{d}\langle L^{(k)} \rangle$$

$$\langle \mathring{\mathbf{S}}^{(\text{aver})} \otimes \mathring{\mathbf{S}}^{(\text{aver})} \rangle = \sum_k \mu^{(k)} \frac{1}{\langle L^{(k)} \rangle} \int_{\langle \mathbf{r}^{(k)} \rangle} \langle \mathring{\mathbf{S}}^{(k)} \otimes \mathring{\mathbf{S}}^{(k)} \rangle \, \mathrm{d}\langle L^{(k)} \rangle \tag{35}$$

where $\langle \hat{\mathbf{S}}^{(k)} \rangle$ and $\langle \mathring{\mathbf{S}}^{(k)} \otimes \mathring{\mathbf{S}}^{(k)} \rangle$ are evaluated by expressions eqns (29) and (30) for each specific reinforcement path $\mathbf{r}^{(k)}(\xi)$.

5. APPLICATIONS OF THE THEORY

Several examples illustrating how the above theory applies to some typical reinforced composite systems are considered in this section.

5.1. *Covariance function and deflection angle*

The main fundamental characteristics (mean and covariance functions) of the stochastic reinforcement path are uniquely defined by the stochastic curve specified in the parametric form eqn (2) or in the form of the canonical expansion eqn (3). Eventually, this direct characterization of the reinforcement path may address specific features of the composite manufacturing; also, some additional *a priori* assumptions are made.

An alternative approach is to specify the mean value and covariance function of the reinforcement path without using its explicit functional definition of the form eqn (2) or (3). Specifically, local fiber waviness (due to crimping, braid angle variation, nonuniform thermal deformations during manufacturing, etc.), is, generally, characterized by two independent parameters—dispersions of the local tangents. The dispersions affect the mean values of elastic constants and their standard deviations. Apparently, this approach opens new possibilities for the effective analysis of elastic properties of composites with locally curved fibers. Indeed, in the existing deterministic approaches, it is necessary to analytically specify exact shape of the fiber waviness (see, Naik and Ganesh, 1992; Naik and Shembekar, 1992; Shembekar and Naik, 1993 as an example). According to the developed stochastic theory, exact characterization of the fiber waviness is not required. Only the integral characteristics, namely dispersions of the local tangent are needed to evaluate elastic properties of the composite. However, additional information concerning the local path of the reinforcement will become necessary if higher moments are utilized in the stochastic analysis. Nevertheless, the developed second-order analysis seems to be sufficiently accurate for most of the practical applications since the variation of the local tangent is usually small.

In the framework of the developed theory, both the above approaches are equivalent and lead to identical final results. This is further illustrated by a simple example of the unidirectional reinforcement along the x_1 direction with local stochastic imperfections in the x_1–x_2 plane. The reinforcement path is characterized as follows:

$$\begin{aligned} x_1 &= \xi \\ x_2 &= \mathring{x}_2(\xi) \end{aligned} \tag{36}$$

where $\mathring{x}_2(\xi)$ is some stochastic centered function. Denoting the angle between local tangent to the reinforcement path and x_1-axis by θ (Fig. 2), the definition of the covariance function provides the following expression:

$$K_{\dot{x}_2\dot{x}_2}(\xi, \xi) = \langle \dot{\mathring{x}}_2(\xi)\dot{\mathring{x}}_2(\xi) \rangle = \left\langle \frac{\mathrm{d}\mathring{x}_2}{\mathrm{d}x_1} \frac{\mathrm{d}\mathring{x}_2}{\mathrm{d}x_1} \right\rangle = \langle \tan^2 \theta \rangle. \tag{37}$$

For the deterministic path, angle θ is constant at each point. Let us consider a set of realizations of some stochastic reinforcement path. Here, "deflection angle" is defined as

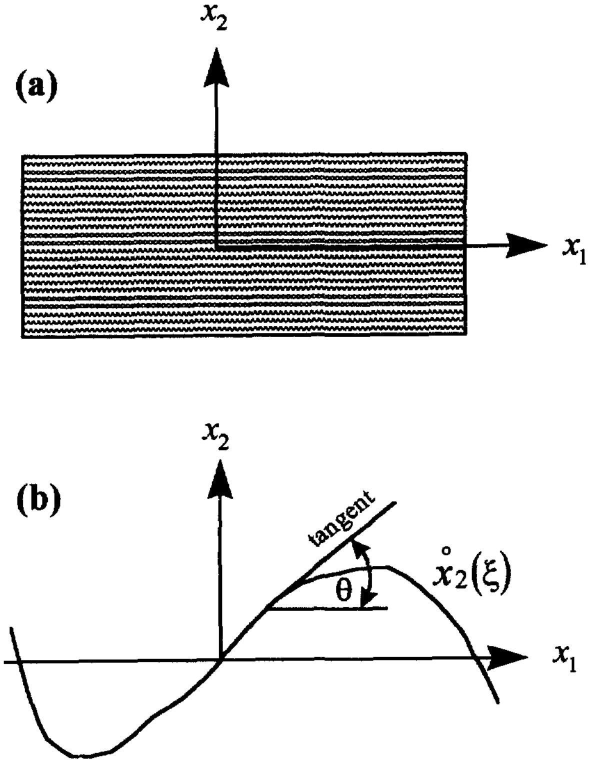

Fig. 2. Unidirectional composite (a) and schematics of local stochastic curvatures along the x_1-axis (b).

the angle between the direction of the perfect reinforcement and local tangent to the realization of the stochastic reinforcement path. Obviously, deflection angle at each point of the stochastic reinforcement path is a random value fluctuating about its mean. If $\theta \ll 1$, then $\tan\theta \approx \theta$, and eqn (37) reduces to

$$K_{\dot{x}_2\dot{x}_2}(\xi,\xi) \approx \sigma_\theta^2 \tag{38}$$

where σ_θ^2 is standard deviation of the deflection angle. After substituting expression (38) in eqns (15) and (17) and applying the above methodology, elastic response of the composite with stochastic reinforcement can be evaluated in the explicit form. Therefore, the theory developed in Sections 2–4 can be utilized even without full description of the stochastic reinforcement path : it is sufficient to only specify the mean reinforcement path and standard deviation of the deflection angle to obtain the mean values and covariance functions of all elastic characteristics of the composite material.

Exactly the same result can be obtained if using explicit form of the stochastic imperfections of the reinforcement path. As an example, these imperfections can be taken as

$$\begin{aligned} x_1 &= \xi \\ x_2 &= A\cos(\kappa\xi - \psi) \end{aligned} \tag{39}$$

where A and ψ are the amplitude and phase of the reinforcement imperfections, respectively, $\kappa = 2\pi/L$, and L is a characteristic length. Assuming that A and/or ψ are random values with some prescribed probabilistic distributions, one obtains the model of the reinforcement path with local stochastic deviations in the x_1–x_2 plane. For example, random-phase model (Chou and Takanashi, 1987) follows from eqn (39) if A is a deterministic value and

$\psi \in [0, 2\pi]$ is a uniformly distributed random value. This corresponds to the stochastic fluctuations with zero mean and with the covariance function

$$K_{x_2 x_2}(\xi, \varsigma) = \frac{A^2}{2} \cos \kappa(\xi - \varsigma). \tag{40}$$

A more general model is obtained from eqn (39) if one assumes that A is a random value with zero mean and standard deviation σ_A. Then, the covariance function of the derivative of the reinforcement path is given by

$$K_{\dot{x}_2 \dot{x}_2}(\xi, \varsigma) = \frac{\sigma_A^2}{2} \kappa^2 \cos \kappa(\xi - \varsigma). \tag{41}$$

The cosine front factor group in eqn (41) has clear geometrical interpretation. Indeed, comparison of expression eqn (41) taken at $\xi = \varsigma$ and expression eqn (37) yields

$$\kappa^2 \sigma_A^2 / 2 = \langle \tan^2 \theta \rangle. \tag{42}$$

Hence, eqn (41) can be written in the form

$$K_{\dot{x}_2 \dot{x}_2}(\xi, \varsigma) = \langle \tan^2 \theta \rangle \cos \kappa(\xi - \varsigma). \tag{43}$$

The most suitable approach of specifying the stochastic reinforcement depends on available technological and experimental information. Apparently, the approach based on prescribing the covariance function of the type eqn (37) is simpler and more suitable since this does not require the knowledge of all the particular details of the reinforcement imperfections.

Our next goal of this section is to show how to specify stochastic imperfections of the general type of 3-D reinforcement using only the standard deviations of the local deflection angle.

Let us specify local orientation of the mean reinforcement path, $\langle \mathbf{r}(\xi) \rangle$, by unit tangent vector, $\mathbf{t}$, and two mutually orthogonal unit vectors, $\mathbf{b}$ and $\mathbf{n}$, both orthogonal to $\mathbf{t}$. Assuming that at each point ξ of the reinforcement path there are some stochastic deflections of the path which are characterized by centered position vector $\mathring{\mathbf{r}}(\xi)$, it is convenient for the purpose of the forthcoming analysis to expand the fluctuations of the position vector into two orthogonal components:

$$\mathring{\mathbf{r}}(\xi) = \mathring{x}_b(\xi)\mathbf{b} + \mathring{x}_n(\xi)\mathbf{n} \tag{44}$$

where $\mathring{x}_b(\xi)$ are random deflections in the plane $\mathbf{t}$–$\mathbf{b}$ and $\mathring{x}_n(\xi)$ are random deflections in the plane $\mathbf{t}$–$\mathbf{n}$. Deflection values in the planes $\mathbf{t}$–$\mathbf{b}$ and $\mathbf{t}-\mathbf{n}$ are designated by A_b and A_n, respectively. Both of them are random values. This separation of the general form of deflections is illustrated in Fig. 3. The triad $\{\mathbf{t}, \mathbf{b}, \mathbf{n}\}$ can be associated, for example, with the principal directions of the unidirectional composite having straight-fiber reinforcement in the direction $\mathbf{t}$. For the case of unidirectional lamina, function $\mathring{x}_b(\xi)$ describes the in-plane waviness and function $\mathring{x}_n(\xi)$ describes the out-of-plane waviness.

By identifying parameter ξ as the arc length of the mean reinforcement path, the covariance of the reinforcement deflections and their derivatives along the mean reinforcement path in the basis $\{\mathbf{t}, \mathbf{b}, \mathbf{n}\}$ are written as the diagonal matrices

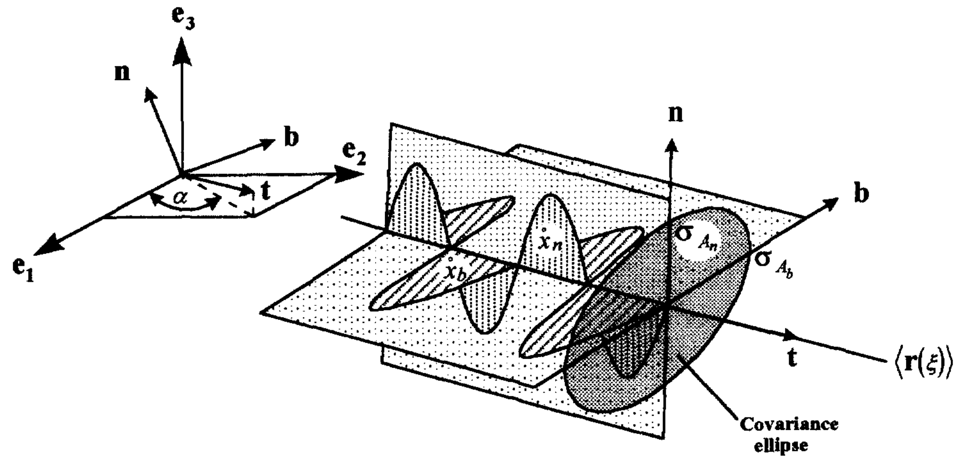

Fig. 3. Local unit vectors, $\{\mathbf{t},\mathbf{b},\mathbf{n}\}$, related to the mean reinforcement path and geometrical explication of decomposition of the stochastic fluctuations of the reinforcement path into two orthogonal components.

$$\hat{\mathbf{K}}_{\mathbf{rr}}^{(t)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{x_b x_b} & 0 \\ 0 & 0 & K_{x_n x_n} \end{bmatrix}, \quad \hat{\mathbf{K}}_{\dot{\mathbf{r}}\dot{\mathbf{r}}}^{(t)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{\dot{x}_b \dot{x}_b} & 0 \\ 0 & 0 & K_{\dot{x}_n \dot{x}_n} \end{bmatrix}. \tag{45}$$

The elements of the covariance matrices eqn (45) are expressed in terms of the standard deviations of the deflection values and the deflection angles

$$K_{x_b x_b}^{(t)} = \langle A_b^2 \rangle, \quad K_{x_n x_n}^{(t)} = \langle A_n^2 \rangle \tag{46}$$

$$K_{\dot{x}_b \dot{x}_b}^{(t)} = \langle \tan^2 \theta_b \rangle, \quad K_{\dot{x}_n \dot{x}_n}^{(t)} = \langle \tan^2 \theta_n \rangle \tag{47}$$

where θ_b and θ_n are the deflection angles in the planes **t**–**b** and **t**–**n**, respectively.

The developed theory is explicitly dependent on the covariance of the derivative, $\hat{\mathbf{K}}_{\dot{\mathbf{r}}\dot{\mathbf{r}}}$, of the reinforcement path. By applying the coordinate transformation to eqn (45), the elements of the covariance matrix referred to the global coordinate system are derived in the form

$$K_{\dot{x}_i \dot{x}_j} = b_i b_j \langle \tan^2 \theta_b \rangle + n_i n_j \langle \tan^2 \theta_n \rangle \tag{48}$$

where $b_i = \mathbf{b}\cdot\mathbf{e}_i$ and $n_i = \mathbf{n}\cdot\mathbf{e}_i$ are the directional cosines of the unit vectors **b** and **n**, respectively. Equation (48) shows that the covariance matrix of the derivative of the stochastic path is expressed in terms of the experimentally measurable characteristics, namely angles θ_b, θ_n and the directional cosines of the assumed straight-fiber reinforcement.

5.2. *Unidirectional composite with curved fibers*

Unidirectional composite with reinforcement along the axis x_1 and random deflection in the plane x_1–x_2 is considered. The position vector for this case is given by

$$\mathbf{r}(\xi) = \{\xi, \mathring{x}_2(\xi), 0\} \tag{49}$$

where $\mathring{x}_2(\xi)$ is an arbitrary centered stationary random function. The tangent vector to the reinforcement path is then given by

$$\dot{\mathbf{r}}(\xi) = \{1, \dot{x}_2(\xi), 0\}. \tag{50}$$

Mean and covariance of the tangent vector are:

$$\langle \dot{\mathbf{r}} \rangle = \{1,0,0\}, \quad \hat{\mathbf{K}}_{\dot{r}\dot{r}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{\dot{x}_2\dot{x}_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{51}$$

where $K_{\dot{x}_2\dot{x}_2} = \langle \mathring{\dot{x}}_2 \mathring{\dot{x}}_2 \rangle$. For the covariance function of form eqn (51), evaluation of the mean of the local basis according to eqn (15) yields

$$\langle \mathbf{e}'_1 \rangle = \{1-\tfrac{1}{2}K_{\dot{x}_2\dot{x}_2}, 0, 0\}, \quad \langle \mathbf{e}'_2 \rangle = \{0, 1-\tfrac{1}{2}K_{\dot{x}_2\dot{x}_2}, 0\}, \quad \langle \mathbf{e}'_3 \rangle = \{0,0,1\}. \tag{52}$$

Next we obtain closed-form expressions for the compliance matrix components following the methodology developed in Sections 2–4. It is customary in mechanics of composites to use contracted notations for the stress, strain, stiffness, and compliance tensors. Specifically, generalized Hooke's law written in the contracted notations has the form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & \text{symm} & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix} \tag{53}$$

where S_{ij} are components of the compliance matrix. Detailed explanation of the relations between tensor notations and contracted notations of stresses, strains, and compliances can be found, for example, in Daniel and Ishai (1994), Bogdanovich and Pastore (1996).

Dropping details of the derivations, the following final result for the mean compliances is obtained from eqn (29):

$$\begin{aligned}
\langle S_{11} \rangle &= \langle e'_{11} \rangle^4 S^0_{11} + K_{\dot{x}_2\dot{x}_2} \langle e'_{11} \rangle^2 (2S^0_{12} + S^0_{66}) \\
\langle S_{12} \rangle &= \langle e'_{11} \rangle^2 \langle e'_{22} \rangle^2 S^0_{12} + K_{\dot{x}_2\dot{x}_2} (\langle e'_{11} \rangle^2 S^0_{11} + \langle e'_{22} \rangle^2 S^0_{22} - \langle e'_{11} \rangle \langle e'_{22} \rangle S^0_{66}) \\
\langle S_{13} \rangle &= \langle e'_{11} \rangle^2 S^0_{13} + K_{\dot{x}_2\dot{x}_2} S^0_{23} \\
\langle S_{22} \rangle &= \langle e'_{22} \rangle^4 S^0_{22} + K_{\dot{x}_2\dot{x}_2} \langle e'_{22} \rangle^2 (2S^0_{12} + S^0_{66}) \\
\langle S_{23} \rangle &= \langle e'_{22} \rangle^2 S^0_{23} + K_{\dot{x}_2\dot{x}_2} S^0_{13} \\
\langle S_{33} \rangle &= S^0_{33} \\
\langle S_{44} \rangle &= \langle e'_{22} \rangle^2 S^0_{44} + K_{\dot{x}_2\dot{x}_2} S^0_{55} \\
\langle S_{55} \rangle &= \langle e'_{11} \rangle^2 S^0_{55} + K_{\dot{x}_2\dot{x}_2} S^0_{44} \\
\langle S_{66} \rangle &= \langle e'_{11} \rangle^2 \langle e'_{22} \rangle^2 S^0_{66} + 2K_{\dot{x}_2\dot{x}_2} [2(\langle e'_{11} \rangle^2 S^0_{11} + \langle e'_{22} \rangle^2 S^0_{22}) - \langle e'_{11} \rangle \langle e'_{22} \rangle (S^0_{66} + 4S^0_{12})].
\end{aligned} \tag{54}$$

Here, S^0_{ij} are compliances of a unidirectional composite with straight fibers. There are additional obvious relations: $S^0_{33} = S^0_{22}$, $S^0_{13} = S^0_{12}$, $S^0_{44} = 2(S^0_{22} - S^0_{23})$, and $S^0_{55} = S^0_{66}$. The matrix form of the compliances S^0_{ij} is written as

$$[S^0_{ij}] = \begin{bmatrix} S^0_{11} & S^0_{12} & S^0_{12} & 0 & 0 & 0 \\ & S^0_{22} & S^0_{23} & 0 & 0 & 0 \\ & & S^0_{22} & 0 & 0 & 0 \\ & & & 2(S^0_{22}-S^0_{23}) & 0 & 0 \\ & \text{symm} & & & S^0_{66} & 0 \\ & & & & & S^0_{66} \end{bmatrix}. \tag{55}$$

For the transversely isotropic material with the plane of isotropy x_2–x_3, there are five independent compliance components and, accordingly, five independent engineering elastic constants; those can be chosen as E^0_{11}, E^0_{22}, G^0_{12}, ν^0_{12}, ν^0_{23}. The rest of the engineering constants are then expressed in terms of the above five: $E^0_{33} = E^0_{22}$, $G^0_{13} = G^0_{12}$, $G^0_{23} = E^0_{22}/[2(1+\nu^0_{23})]$.

By using eqn (38), $K_{\dot{x}_2\dot{x}_2}$ is further expressed in terms of the standard deviation of the deflection angle θ (see Fig. 2), using the assumption that θ is small:

$$K_{\dot{x}_2\dot{x}_2} = \langle \tan^2\theta \rangle \approx \langle \theta^2 \rangle = \sigma^2_\theta$$

Then, using this relation and substituting expression $S^0_{ii} = 1/E^0_{ii}$, $S'_{ij} = -(\nu^0_{ij}/E^0_{ij})$ (i,j = 1, 2, 3), $S^0_{44} = 1/G^0_{23}$, $S^0_{55} = 1/G^0_{13}$, $S^0_{66} = 1/G^0_{12}$ into eqn (54), one obtains the following expressions for the mean values of elastic moduli:

$$\frac{\langle E_{11} \rangle}{E^0_{11}} = 1-\sigma^2_\theta\left[\frac{E^0_{11}}{G^0_{12}} - 2(1+\nu^0_{12})\right] \tag{56}$$

$$\frac{\langle E_{22} \rangle}{E^0_{22}} = 1-\sigma^2_\theta\left[\frac{E^0_{22}}{G^0_{12}} - 2\left(1+\nu^0_{12}\frac{E^0_{22}}{E^0_{11}}\right)\right] \tag{57}$$

$$\frac{\langle E_{33} \rangle}{E^0_{33}} = 1 \tag{58}$$

shear moduli:

$$\frac{\langle G_{23} \rangle}{G^0_{23}} = 1-\sigma^2_\theta\left(\frac{G^0_{23}}{G^0_{13}} - 1\right) \tag{59}$$

$$\frac{\langle G_{13} \rangle}{G^0_{13}} = 1-\sigma^2_\theta\left(\frac{G^0_{13}}{G^0_{23}} - 1\right) \tag{60}$$

$$\frac{\langle G_{12} \rangle}{G^0_{12}} = 1-4\sigma^2_\theta\left[G^0_{12}\left(\frac{1+2\nu^0_{12}}{E^0_{11}} + \frac{1}{E^0_{22}}\right) - 1\right] \tag{61}$$

and Poisson's ratios:

$$\langle \nu_{12} \rangle = \nu^0_{12} + \sigma^2_\theta\left[2(\nu^0_{12})^2 - 1 + \frac{E^0_{11}}{G^0_{12}}\left(1-\nu^0_{12} - \frac{G^0_{12}}{E^0_{22}}\right)\right] \tag{62}$$

$$\langle \nu_{13} \rangle = \nu^0_{13} + \sigma^2_\theta\left[\nu^0_{13}(1+2\nu^0_{12}) + \frac{E^0_{11}}{E^0_{22}}\left(\nu^0_{23} - \nu^0_{13}\frac{E^0_{22}}{G^0_{12}}\right)\right] \tag{63}$$

$$\langle \nu_{23} \rangle = \nu_{23}^0 + \sigma_\theta^2 \left[\nu_{23}^0 \left(1 + 2\nu_{21}^0 - \frac{E_{22}^0}{G_{12}^0} \right) + \nu_{31}^0 \right]. \tag{64}$$

Here, $\langle E_{ii} \rangle$, $\langle G_{ij} \rangle$, and $\langle \nu_{ij} \rangle$ are the mean values of the elastic moduli, shear moduli, and Poisson's ratios, respectively. All small terms of the order higher than $o(\sigma_\theta^4)$ have been disregarded in the eqns (56)–(64).

For the composites with $E_{11}^0/G_{12}^0 \gg 1$, the relation eqn (56) reduces to $\langle E_{11} \rangle / E_{11}^0 \approx 1 - \sigma_\theta^2 E_{11}^0/G_{12}^0$. This expression for the mean longitudinal elastic modulus coincides with the result obtained by Bolotin (1966). He derived this result considering reinforced layers with small initial waviness forming a homogeneous stochastic field. Bolotin employed the spectral theory of random functions for the analysis of layered media.

Cox (1995) analyzed the effect of local fiber waviness using the technique of direct formal averaging of elastic constants of a homogeneous anisotropic material. He considered an axially loaded wavy tow as a sequence of misoriented unidirectional composite segments bearing equal stress in the load direction. The spatially averaged longitudinal elastic modulus $\langle E_{11} \rangle$ of such a tow is given by

$$\langle E_{11} \rangle = \left[\int_{-\infty}^{\infty} \frac{f(\theta)\, \mathrm{d}\theta}{E_{11}(\theta)} \right]^{-1} \tag{65}$$

where $E_{11}(\theta)$ is elastic modulus of unidirectional composite under the load oriented at angle θ to the fiber direction x_1. Assuming that $f(\theta)$ is Gaussian distribution, closed-form expression for $\langle E_{11} \rangle$ is obtained after performing the integration in eqn (65). The resulting expression (see also eqn (23d) of Xu *et al.*, 1995) is exactly the same as eqn (56). Note that Cox (1995) obtained closed-form solution only for the longitudinal elastic modulus, while eqns (56)–(64) define all engineering constants. Besides that, eqns (56)–(64) do not assume any specific form of the distribution law of the deflection angle.

Numerical results for the elastic moduli, shear moduli and Poisson's ratios are shown in Fig. 4. The results illustrate effect of the random fiber deflections for the unidirectional composite with $E_f = 76$ GPa, $E_m = 4$ GPa, $\nu_f = 0.28$, $\nu_m = 0.38$ and fiber volume fraction $V_f = 0.6$. Two analytical approaches have been used: eqns (54) and closed-form solution eqns (56)–(64). It is seen that for small standard deviations both approaches give very close results. However, eqns (54) are applicable for a broader range of σ_θ than eqns (56)–(64). As seen in Fig. 4a, most severe effect is obtained for the mean value of the longitudinal elastic modulus, $\langle E_{11} \rangle$. Particularly, at $\sigma_\theta = 25^\circ$ its drop is about 50%. Less significant drop is obtained for the mean value of the transverse elastic modulus E_{22} in the plane of the reinforcement deflection. Naturally, mean value of E_{33} is not affected by the reinforcement deflection. As is seen from eqns (59)–(61) and Fig. 4b, all three shear moduli are affected by the fiber curvatures. The values of $\langle G_{12} \rangle$ are significantly higher than G_{12}^0, which corresponds to well-known effect of increasing shear resistance with higher fiber waviness. The effect of fiber curvatures on the other two shear moduli is negligible. Finally, as Fig. 4c shows, the curvatures significantly affect the mean values of all three Poisson's ratios, especially $\langle \nu_{12} \rangle$, which is increased by 60% at $\sigma_\theta = 25^\circ$.

5.3. *Helically wound composites with curved reinforcement*

Consider a cylindrical shell of radius R with a helical reinforcement, Fig. 5. A "perfect" helical reinforcement path with the pitch length h can be represented as

$$x_1(z) = R\cos\left(\frac{2\pi}{h} z\right)$$

$$x_2(z) = R\sin\left(\frac{2\pi}{h} z\right)$$

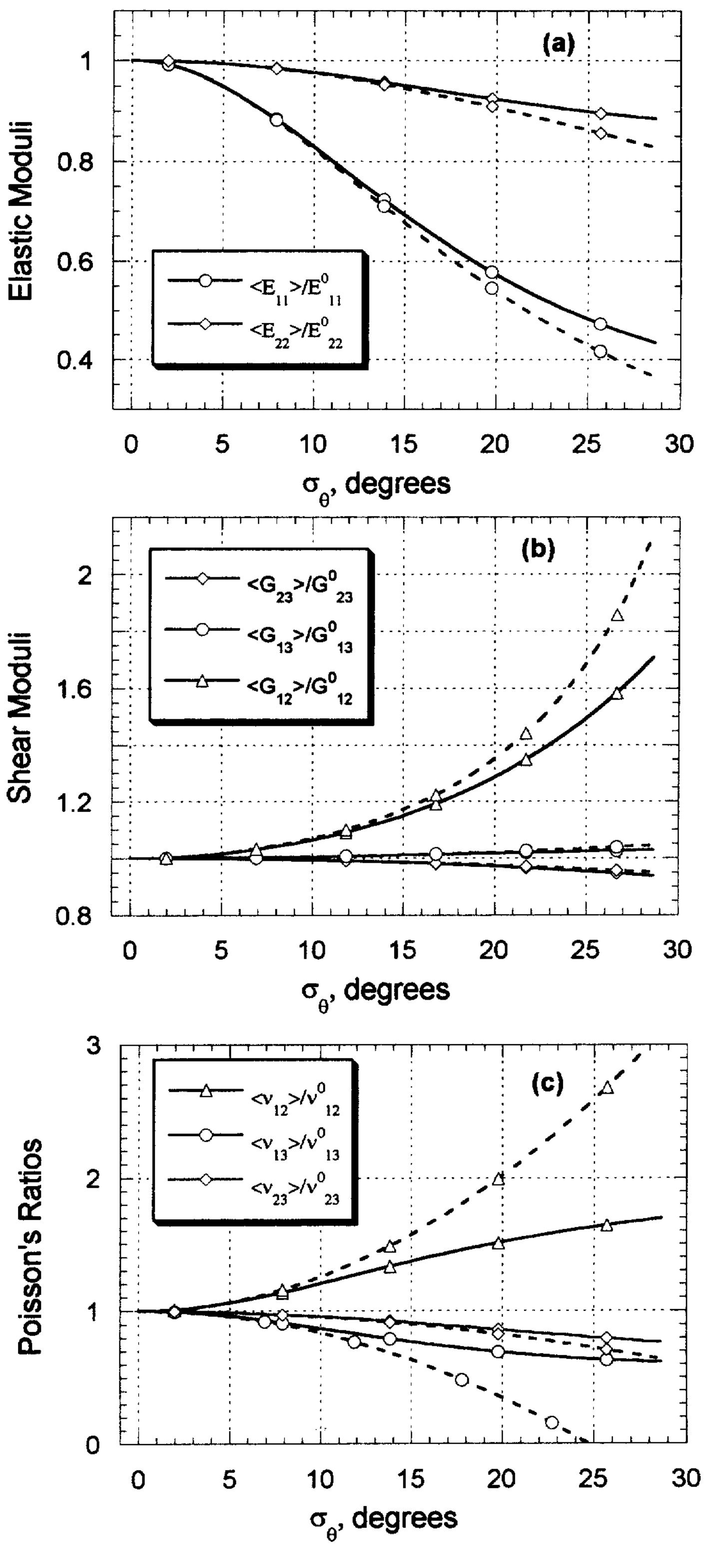

Fig. 4. Elastic characteristics of the unidirectional composite vs the standard deviation, σ_θ, of the deflection angle. Solid lines correspond to eqn (54) and dashed lines correspond to eqns (56)–(64).

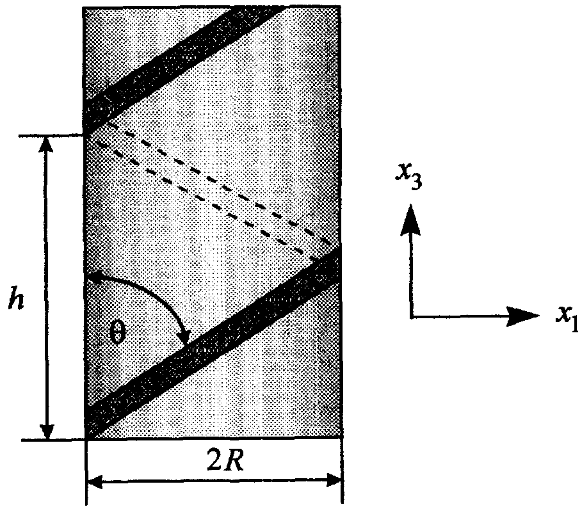

Fig. 5. Geometrical characteristics of the helix reinforcement path.

$$x_3(z) = z \tag{66}$$

where $0 \leqslant z \leqslant h$. Helix angle θ is related to the parameters R and h through the relation $\tan\theta = 2\pi R/h$. It is convenient to use the arc length for the parametric representation of the helical reinforcement path. The arc length, ξ, of the helix specified by eqn (66) is defined as

$$\xi = \int_0^z [\dot{x}^2(\varsigma)+\dot{y}^2(\varsigma)+\dot{z}^2(\varsigma)]^{1/2}\,\mathrm{d}\varsigma = \frac{z}{\cos\theta}. \tag{67}$$

Hence, after changing variable $z = \xi\cos\theta$, equation of the helix eqn (66) transforms to the parametric form

$$\begin{aligned} x_1(\xi) &= R\cos(\omega\xi) \\ x_2(\xi) &= R\sin(\omega\xi) \\ x_3(\xi) &= \xi\cos\theta \end{aligned} \tag{68}$$

where $\omega = \sin\theta/R$ and $0 \leqslant \xi \leqslant 2\pi/\omega$. Variation of the arc length over the interval $\xi \in [0, 2\pi/\omega]$ corresponds to one complete turn of the helix. Note that the following normalization relation is satisfied $|\dot{\mathbf{r}}(\xi)| \equiv 1$; this relation is the consequence of using the arc length as a parameter.

Helix angle θ may have random local deviations due to various technological effects. Cylindrical structures with perfect and imperfect helical reinforcements are shown in Fig. 6. According to eqn (68), local deviation of the helix angle, $\delta\theta$, results in the following deflection of the reinforcement path:

$$\delta x_1 = -\xi\cos\theta\sin(\omega\xi)\delta\theta, \quad \delta x_2 = \xi\cos\theta\cos(\omega\xi)\delta\theta, \quad \delta x_3 = -\xi\sin\theta\,\delta\theta. \tag{69}$$

Considering $\delta\theta$ as a random variable with zero mean value, $\langle\delta\theta\rangle = 0$, and given standard deviation σ_θ, the stochastically imperfect helix reinforcement is characterized as follows:

$$\mathbf{r}(\xi) = \langle\mathbf{r}(\xi)\rangle + \mathring{\mathbf{r}}(\xi) \tag{70}$$

where

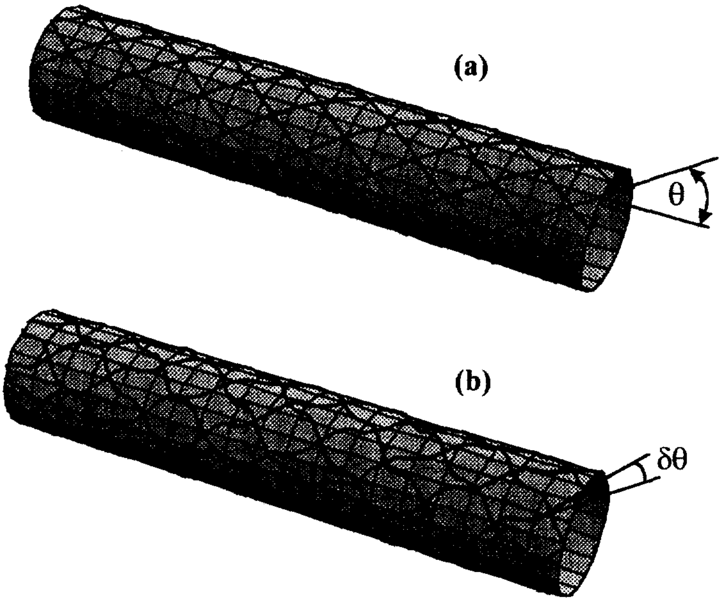

Fig. 6. Cylindrical wound composite with perfect (a) and imperfect (b) helical reinforcement.

$$\langle \mathbf{r}(\xi) \rangle = \{R\cos(\omega\xi), \quad R\sin(\omega\xi), \quad \xi\cos\omega\} \tag{71}$$

is the mean reinforcement path representing "perfect" helix and

$$\mathring{\mathbf{r}}(\xi) = \{-\xi\cos\theta\sin(\omega\xi)\,\delta\theta, \quad \xi\cos\theta\cos(\omega\xi)\,\delta\theta, \quad -\xi\sin\theta\,\delta\theta\} \tag{72}$$

is the local random deflection. The derivative of the reinforcement deflection with respect to the arc length is calculated as

$$\dot{\mathring{\mathbf{r}}}(\xi) = \frac{\partial \mathring{\mathbf{r}}(\xi)}{\partial \xi} = \{s_1\,\delta\theta, s_2\,\delta\theta, s_3\,\delta\theta\} \tag{73}$$

where

$$\begin{aligned} s_1 &= [\sin(\omega\xi) - \omega\xi\cos(\omega\xi)]\cos\theta \\ s_2 &= [\cos(\omega\xi) - \omega\xi\sin(\omega\xi)]\cos\theta \\ s_3 &= -\sin\theta. \end{aligned} \tag{74}$$

Using eqn (73), the full covariance matrix of the derivatives of the local deflections is then obtained as

$$\hat{\mathbf{K}}_{\dot{r}\dot{r}}(\xi,\xi) = \sigma_\theta^2 \begin{bmatrix} s_1^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & s_2^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & s_3^2 \end{bmatrix}. \tag{75}$$

Equations (71), (74), and (75) are sufficient for applying the developed theory for composite materials with the helical reinforcements. Vector $\boldsymbol{\lambda}$ entering in eqns (10)–(12) was chosen as $\boldsymbol{\lambda} = \langle \dot{\mathbf{r}}(\xi) \rangle \times \langle \ddot{\mathbf{r}}(\xi) \rangle$ to ensure that condition $\boldsymbol{\lambda} \times \langle \dot{\mathbf{r}}(\xi) \rangle \neq 0$ is satisfied at every point along the mean reinforcement path.

Numerical analysis based on the compliance averaging eqns (29), (30) and (35) was carried out. The effect of the standard deviation of the deflection angle on the elastic characteristics of the wound composite with helical reinforcement is illustrated in Figs 7–10. The results were obtained for Graphite/Epoxy composite with the following elastic characteristics of the fibers (Fiberite®, 1995) : $E_f = 230$ GPa, $\nu_f = 0.3$ and matrix : $E_m = 3.45$ GPa, $\nu_m = 0.35$. Total fiber volume fraction is 0.6. The helix angle was taken $\theta = \pm 20°$.

From the results presented in Figs 7–10, one can conclude that local random reinforcement deflections cause significant change of both the mean values and standard deviations of the elastic constants. Particularly, at a very moderate deflection angle, $\sigma_\theta = 10°$, the drop of the mean value of the elastic modulus E_{33} is 45% and its variation about the mean is $\approx 15\%$ (Fig. 7c). The dependencies of the mean values of the elastic moduli E_{11} and E_{22} on the standard deviation of the deflection angle are much less conspicuous, as it follows from Figs 7a and 7b. Namely, the drop of the mean value of E_{11} is only 2.6% and its variation about the mean is 4% (Fig. 7a) ; the drop of the mean value of E_{22} is 4% and its variation about the mean is $\approx 10\%$ (Fig. 7b). In fact, the standard deviation of the moduli E_{11} and E_{22} overlap the drop of their mean values.

Shear moduli show different behavior: the values of $\langle G_{23} \rangle$ and $\langle G_{13} \rangle$ increase when increasing the standard deviation of the deflection angle. According to Figs 8a and 8b, the mean values of G_{23} and G_{13} increase up to 24 and 21%, respectively, at $\sigma_\theta = 10°$ compared to their values for the perfect helical reinforcement. Variation of G_{23} and G_{13} is about 20% at $\sigma_\theta = 10°$. The mean value of shear moduli G_{12} practically does not change with the reinforcement waviness, but its variation reaches 7.6% at $\sigma_\theta = 10°$ (Fig. 8c).

The Poisson's ratios are the most sensitive to the local waviness of the reinforcement. Indeed, according to Figs 9a and 9b, the mean values and variations of ν_{23} and ν_{13} are $100 \pm 38\%$ and $92 \pm 32\%$, respectively, at $\sigma_\theta = 10°$. The dependencies of the mean value and standard deviation of ν_{12} are nonmonotone functions of the standard deviation of the deflection angle (Fig. 9c). At small σ_θ, the values of $\langle \nu_{12} \rangle$ and $\sigma_{\nu_{12}}$ decrease reaching minimum at $\sigma_\theta \approx 7°$. At $\sigma_\theta > 7°$, the mean value of ν_{12} and its variation increase with increasing the standard deviation of the deflection angle.

Figures 10–12 show the dependencies of the elastic characteristics on the helix angle θ. The results for the perfect helical reinforcement were obtained using the developed theory at $\sigma_\theta = 0$. It can be noted that the same result can be obtained using analytical closed-form solution given by eqns (13) and (14) of Byun and Chou (1995). The results for the imperfect helical reinforcement with the standard deviation of helix angle $\sigma_\theta = 10°$ are also presented in Figs 10–12. The upper (e.g. $\langle E_{11} \rangle + \sigma_{E_{11}}$) and lower (e.g. $\langle E_{11} \rangle - \sigma_{E_{11}}$) bounds of the standard deviations of the elastic characteristics are plotted by dashed lines. From these figures one can conclude that local random waviness cause significant change of both the mean values and standard deviations of the elastic constants over the entire interval of helix angle $0° < \theta < 90°$.

Two significant features of the elastic response of helical wound composites with random reinforcement waviness should be pointed out. The first is related to the effect of the helix angle deflection on the elastic symmetry. For the perfect helical reinforcement, the elastic constants having indices 23 and 13 are identical : $E_{23} \equiv E_{13}$, $G_{23} \equiv G_{13}$, and $\nu_{23} \equiv \nu_{13}$. However, certain difference between the elastic characteristics having indices 23 and 13 is revealed when accounting for the random waviness, i.e. when $\sigma_\theta \neq 0$. This effect can be explained by the fact that the reinforcement deviations are not commutative with respect to the indices 1 and 2. Indeed, according to the relations eqn (74), the covariance matrix eqn (75) changes under the cyclic permutation of the indices 1 and 2.

Another peculiarity can be observed from the comparison of the dependencies shown in Figs 10–12 with the corresponding dependencies presented in Fig. 4. Namely, the local reinforcement waviness causes more severe effect on the elastic characteristics of the considered helical wound composite in comparison with the case of a unidirectional composite. This effect can be explained by comparing the covariance matrices eqn (51) for the unidirectional composite and eqn (75) for the helical wound composite. In the case of the unidirectional composite, the covariance matrix has only one non-zero element. Hence, variation of the elastic properties is the result of the variation of the deflection angle only in one plane, specifically, in the plane x–z. For the helical wound composite, the covariance matrix eqn (75) has a full form, and, hence, variation of the elastic properties is the result

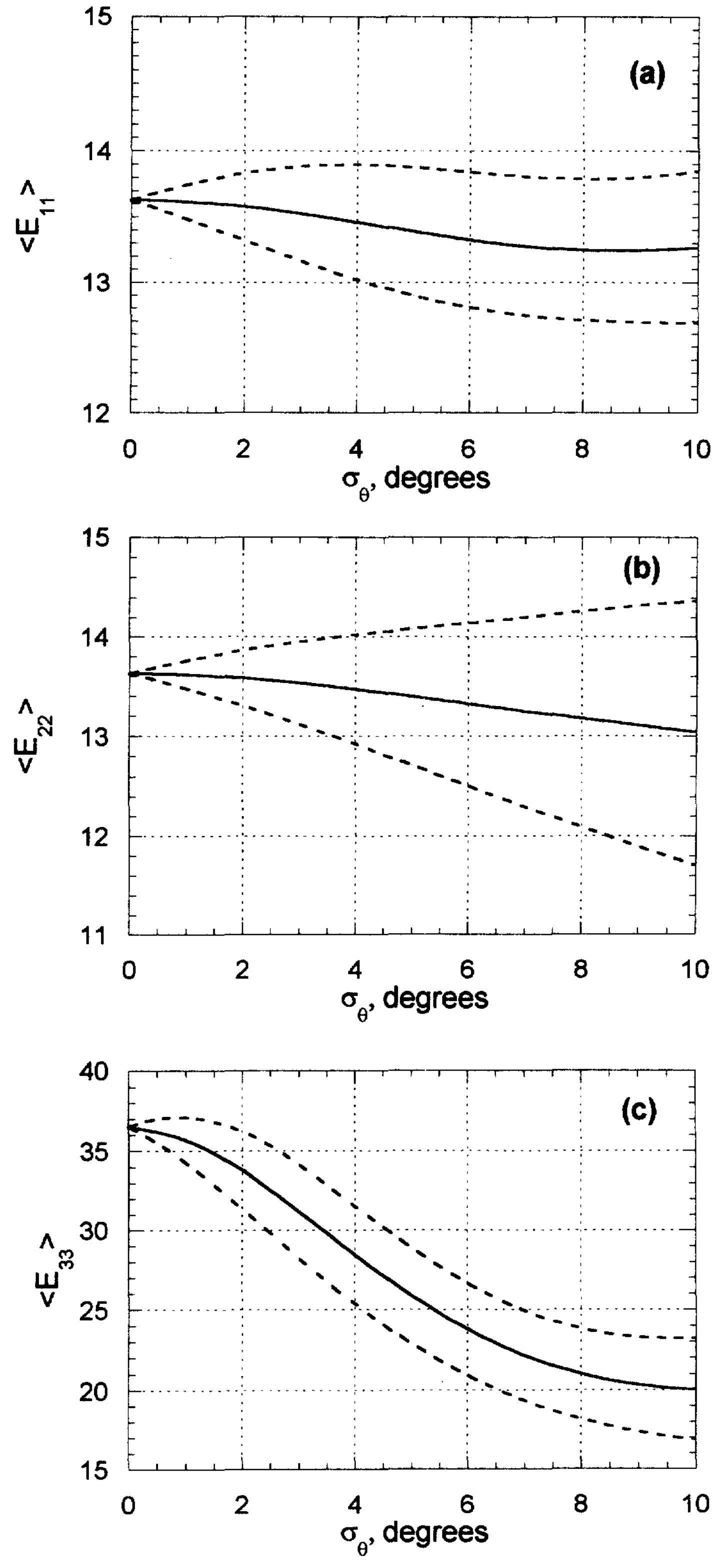

Fig. 7. Dependencies of the elastic moduli of helical wound composite on the standard deviation, σ_θ, of the deflection angle. Solid lines correspond to the mean values of the elastic moduli and the dashed lines are the bounds of their standard deviations.

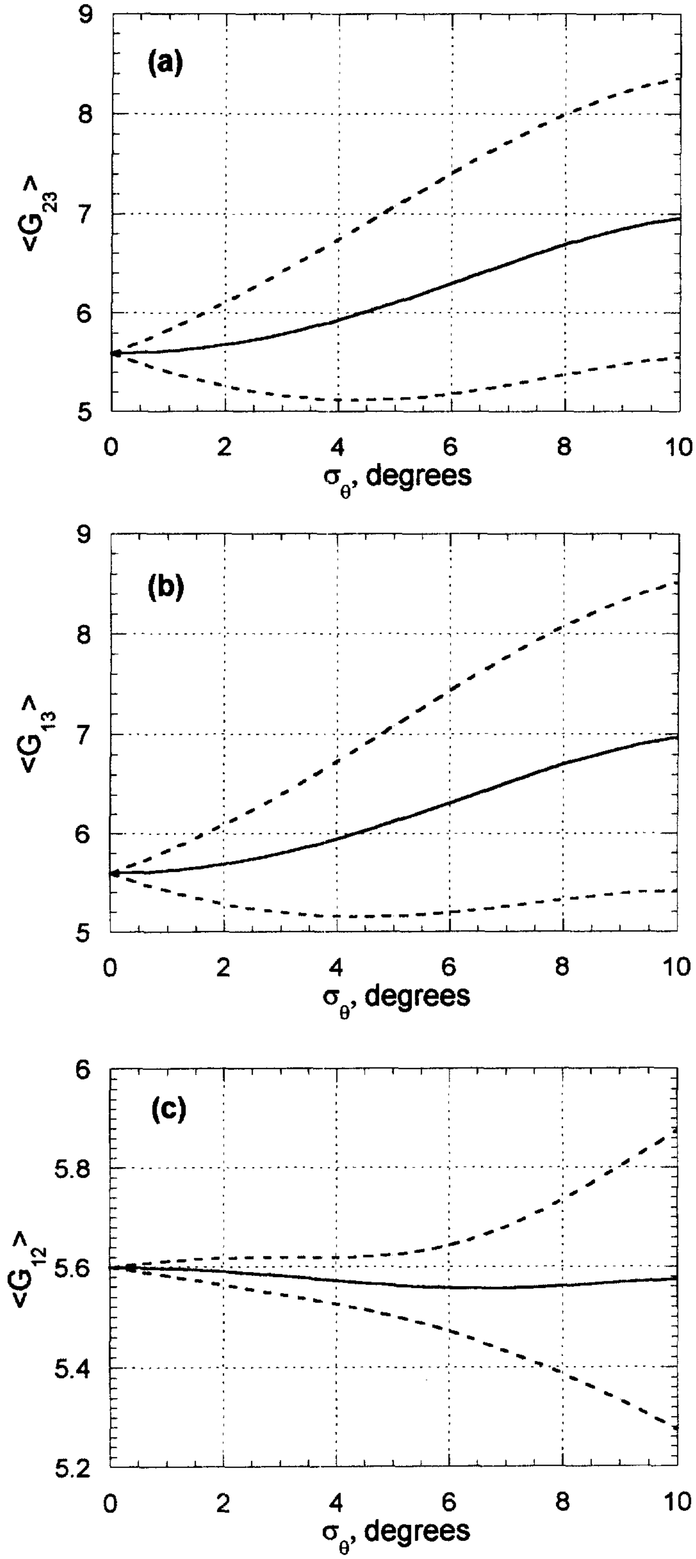

Fig. 8. Dependencies of the shear moduli of helical wound composite on the standard deviation, σ_θ of the deflection angle. Solid lines correspond to the mean values of the shear moduli and the dashed lines are bounds of their standard deviations.

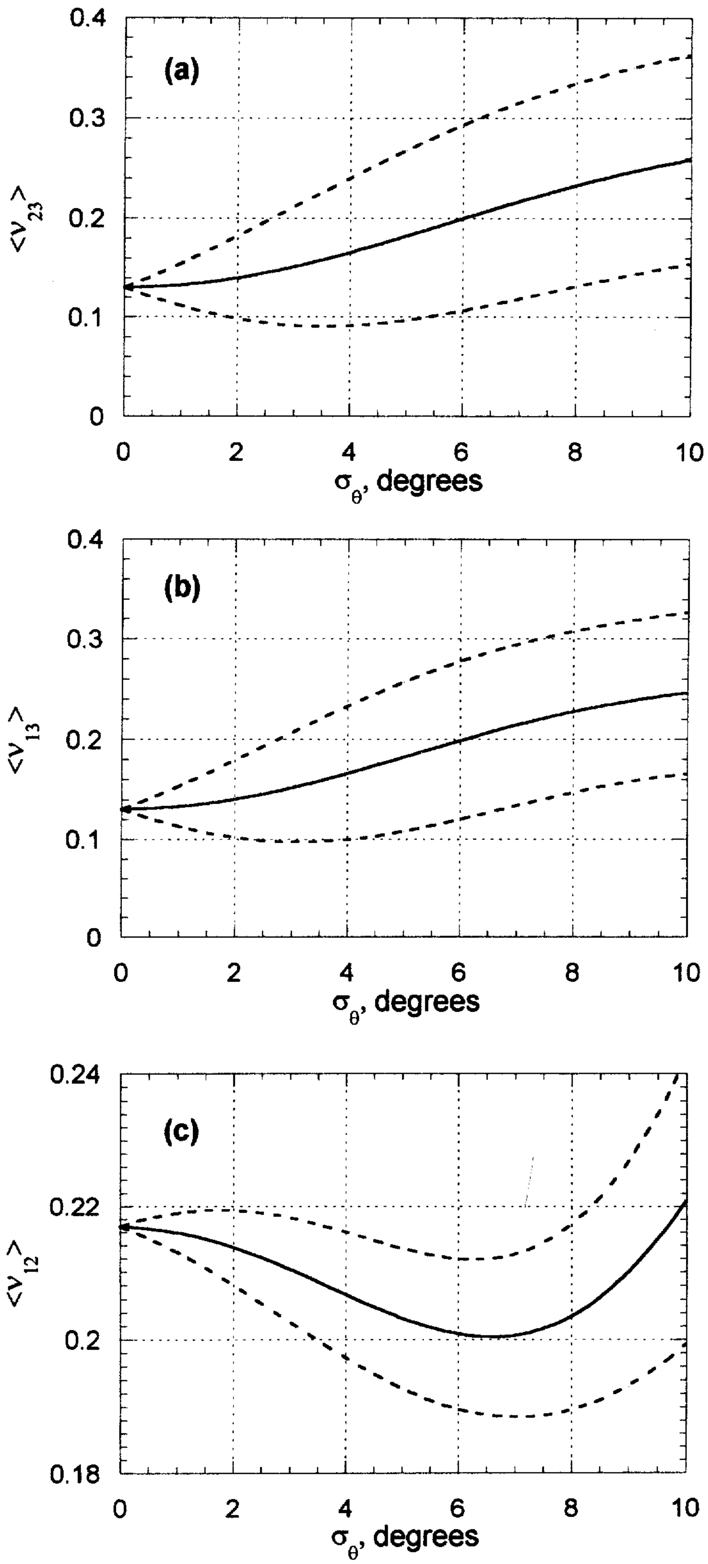

Fig. 9. Dependencies of the Poisson's ratios of helical wound composite on the standard deviation, σ_θ, of the deflection angle. Solid lines correspond to the mean values of the Poisson's ratios and the dashed lines are the bounds of their standard deviations.

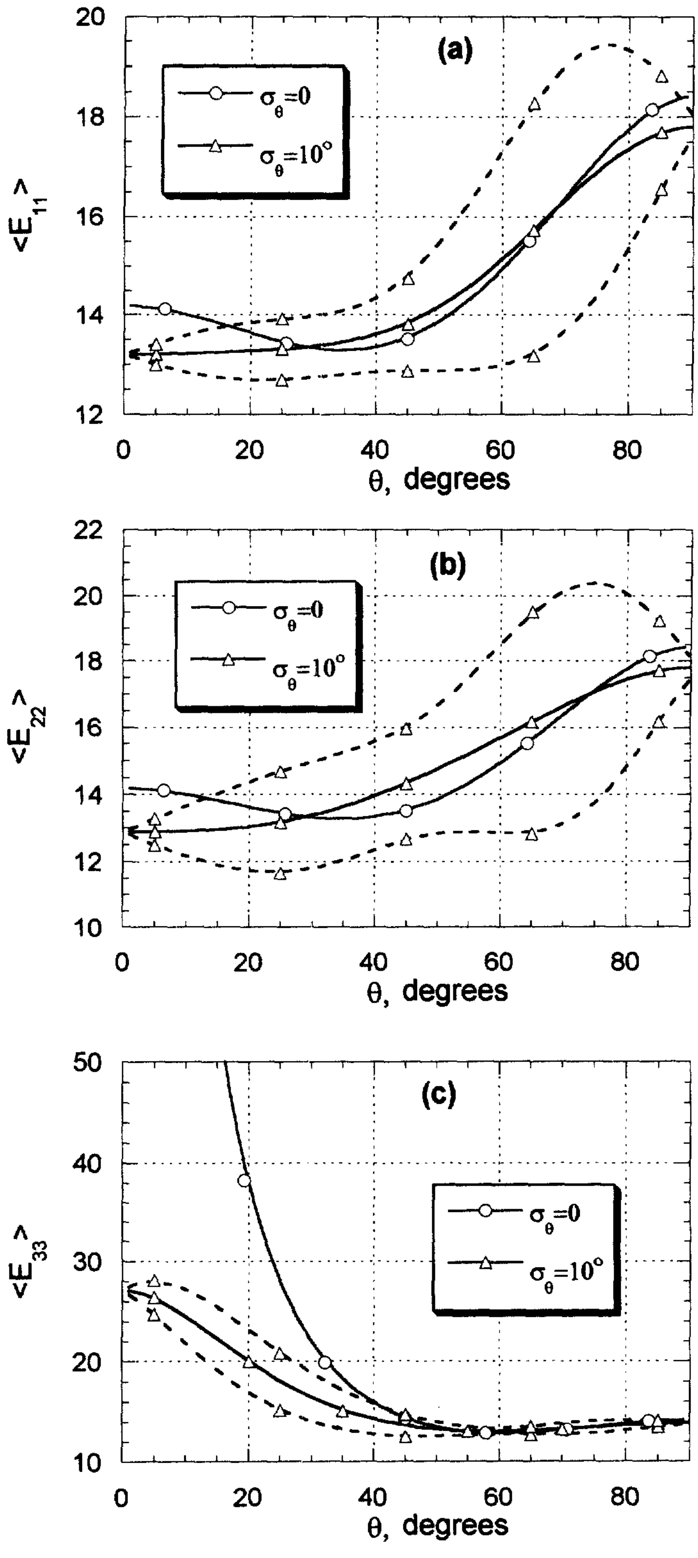

Fig. 10. Dependencies of the elastic moduli of wound composite on the helix angle θ. Dashed lines are the standard deviation bounds of the elastic moduli at $\sigma_\theta = 10°$.

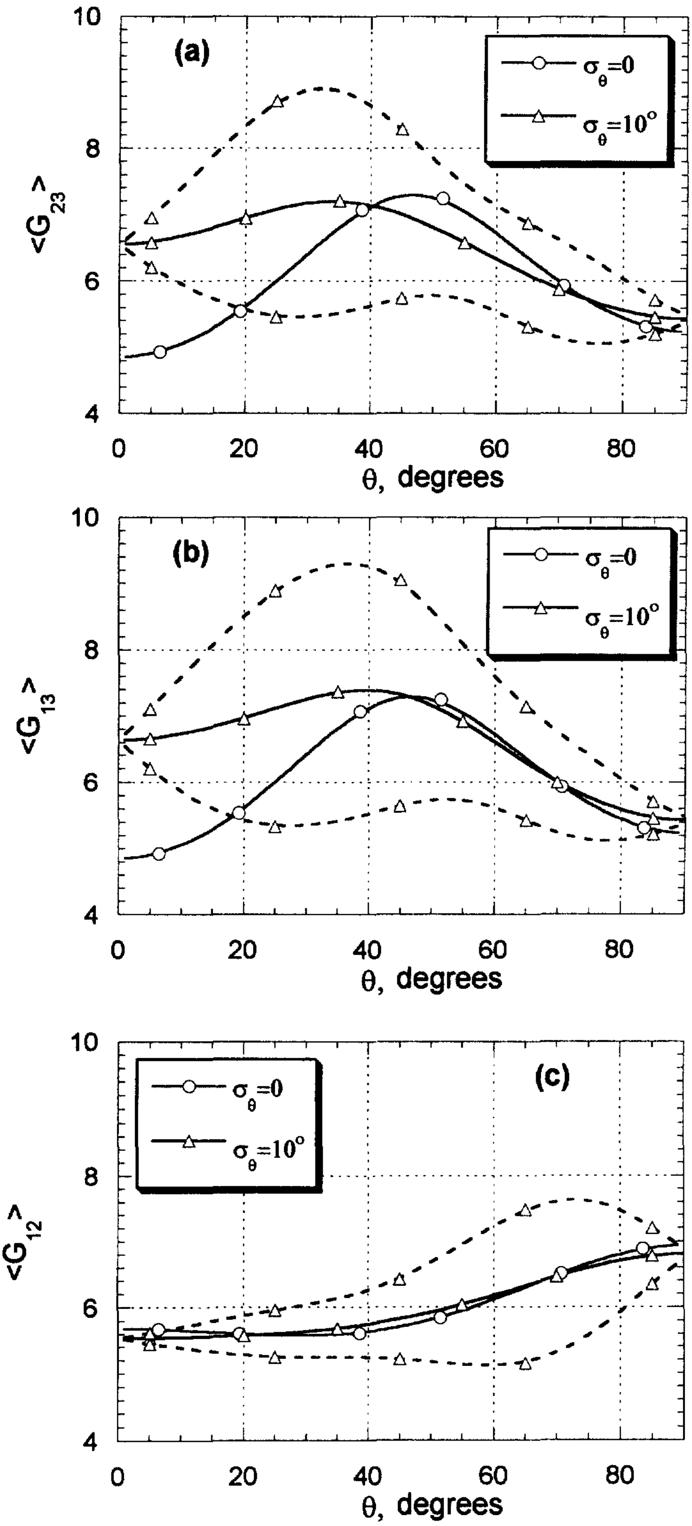

Fig. 11. Dependencies of the shear moduli of wound composite on the helix angle θ. Dashed lines are the standard deviation bounds of the shear moduli at $\sigma_\theta = 10^\circ$.

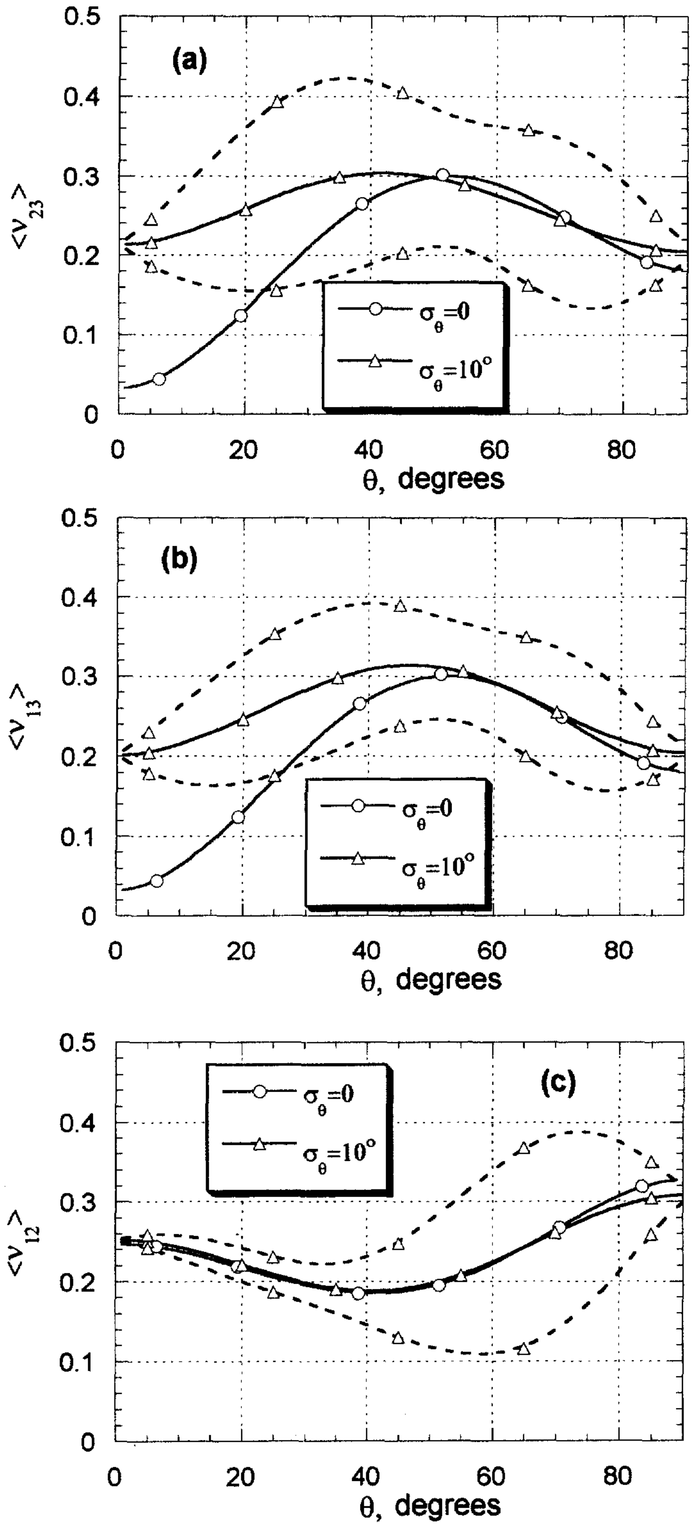

Fig. 12. Dependencies of the Poisson's ratios of wound composite on the helix angle θ. Dashed lines are the standard deviation bounds of the Poisson's ratios at $\sigma_\theta = 10°$.

of the variation of the reinforcement path in all three planes: x–y, x–z, and y–z. Apparently, variation of the reinforcement path in all three planes causes more significant change of the elastic characteristics than variation of the reinforcement path in one plane only.

6. CONCLUSIONS

- A novel stochastic theory of composites with continuous, multidirectional spatial reinforcements having stochastic waviness, has been developed. The theory predicts the elastic response using stochastic generalization of the spatial stiffness/compliance averaging approaches. The input information required is limited to the mean and standard deviations of the stochastic reinforcement paths.
- Existing approaches for predicting elastic response of unidirectional composites with curved fibers can be obtained from the developed theory as particular cases.
- Closed-form solutions for composites with unidirectional stochastically curved fiber reinforcements have been obtained. The numerical examples illustrate that the developed theory can be readily applied for the analysis of various classes of composites with multidirectional spatial reinforcements, taking into account random reinforcement waviness.
- The developed stochastic theory provides a powerful analytical tool for the elastic analysis of various types of imperfect composite systems with randomly curved reinforcement.

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REFERENCES

Bogdanovich, A. E. and Pastore, C. M. (1996) *Mechanics of Textile and Laminated Composites*. Chapman and Hall, London.

Bolotin, V. V. (1966) Theory of a reinforced layered medium with random initial irregularities. *Polymer Mechanics* **2**(1), 11–19.

Byun, J. H. and Chou, T.-W. (1995) Effect of yarn twist on the elastic property of composites. *Proceedings of the Tenth International Conference on Composite Materials*, Volume IV: Characterization and ceramic matrix composites. Whistler, B.C., Canada, 14–18 August 1995, pp. 293–299. Woodhead Publishing Limited.

Chen, J., McBride, T. M. and Sanchez, S. B. (1996) Sensitivity of mechanical properties to braider misalignment in triaxial braid composites. *Proceedings of the American Society for Composites, Eleventh Technical Conference*, 7–9 October 1996, Atlanta Georgia, pp. 1007–1015. TECHNOMIC Publishing Co., Lancaster.

Chou, T.-W. and Takahashi, K. (1987) Non-linear elastic behaviour of flexible fibre composites. *Composites* **18**(1), 25–34.

Clyburn, C. E., III (1993) The enhancement of yarns in graphite–epoxy composites for imaging with conventional radiography. *Proceedings of the American Society for Composites, Eighth Technical Conference*, 19–21 October 1996, Cleveland, Ohio, pp. 337–342. TECHNOMIC Publishing Co., Lancaster.

Cook, J. (1968) The elastic constants of an isotropic matrix reinforced with imperfectly oriented fibers. *Brit. J. Appl. Phys. (J. Phys. D), Series* 2 **1**, 799–812.

Cox, B. (1995) *Failure Models for Textile Composites*. NASA Contractor Report 4686.

Cox, H. L. (1952) The elasticity and strength of paper and other fibrous materials. *Brit. J. Appl. Phys.* **3**, 72–79.

Daniel, I. M. and Ishai, O. (1994) *Engineering Mechanics of Composite Materials*. Oxford University Press, New York.

ICI Fiberite® Materials Handbook. ICI Fiberite Technology Group, Tempe, AZ, 1995.

Gowayed, Y. A., Pastore, C. and Howarth, C. S. (1996) Modification and application of a unit cell continuum model to predict the elastic properties of textile composites. *Composites: Part A* **27A**(2), 149–155.

Kregers, A. F. (1979) Deformation properties of composite materials with spatial reinforcement. *Mechanics of Composite Materials* **5**, 790–793.

Kregers, A. F. (1982) Structural model for deformation of spatially reinforced composite. *Mechanics of Composite Materials* **1**, 14–22.

Kreger, A. F. and Melbardis, Yu. G. (1978) Determination of the deformability of three-dimensionally reinforced composites by the stiffness averaging method. *Polymer Mechanics* **14**(1), 3–8.

Kuo, C.-M., Takahashi, K. and Chou, T.-W. (1988) Effect of fiber waviness on the nonlinear elastic behavior of flexible composites. *Journal of Composite Materials* **22**(11), 1004–1025.

Liu, D. and Xu, L. (1995) Effects of fiber waviness and bonding condition on composite performance. *Proceedings of the Tenth International Conference on Composite Materials*, Volume IV: Characterization and ceramic matrix composites, Whistler, B.C., Canada, 14–18 August 1995, pp. 277–284. Woodhead Publishing Limited.

Naik, N. K. and Ganesh, V. K. (1992). Prediction of on-axes elastic properties of plain weave fabric composites. *Composites Science and Technology* **45**, 135–152.

Naik, N. K. and Shembekar, P. S. (1992) Elastic behavior of woven fabric composites: I—lamina analysis. *Journal of Composite Materials* **26**(15), 2196–2225.

Pastore, C. M. (1993) Quantification of processing artifacts in textile composites. *Composites Manufacturing* **4**(4), 217–226.

Pastore, C. M. and Gowayed, Y. A. (1994) A self-consistent fabric geometry model: modification and application of a fabric geometry model to predict the elastic properties of textile composites. *Journal of Composites Technology & Research* **16**(1), 32–36.

Pugachev, V. S. (1965) *Theory of Random Functions and its Application to Control Problems.* Pergamon Press, London. (Translation from Russian, Nauka, Moscow, 1962.)

Shembekar, P. S. and Naik, N. K. (1993) Elastic analysis of woven fabric laminates: part I. Off-axis loading. *Journal of Composites Technology and Research* **15**, 23–33.

Sun, F., Kimpara, I. and Kageyama, K. (1995) Calculation models and constitutive equations of composites with two or three dimensional wavy fibers. *Proceedings of the Tenth International Conference on Composite Materials*, Volume IV: Characterization and ceramic matrix composites, Whistler, B.C., Canada, 14–18 August 1995, pp. 325–332. Woodhead Publishing Limited.

Xu, J., Cox, B. N., McGlockton, M. A. and Carter, W. C. (1995) A binary model of textile composites—II. The elastic regime. *Acta Metall. Mater.* **43**(9), 3511–3535.

Yushanov, S. P. and Bogdanovich, A. E. (1998) Analytical probabilistic modeling of initial failure and reliability of laminated composite structures. *International Journal of Solids and Structures* **35**(7–8), 665–685.

APPENDIX A. COEFFICIENTS OF THE LOCAL BASIS EXPANSION

Diadic technique

Diadic notation is used in the following calculations. Namely, the kth-order tensor $\hat{\mathbf{X}}$ is represented in the diadic form as

$$\hat{\mathbf{X}} = X_{i_1 i_2 \ldots i_k} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \ldots \otimes \mathbf{e}_{i_k} \tag{A1}$$

where $X_{i_1 i_2 \ldots i_k}$ are the components of the tensor $\hat{\mathbf{X}}$, and $\mathbf{e}_i$ are the basis vectors of the global coordinate system. Symbol $\otimes$ denotes outer tensor product. As usual, the summation convention is used: every letter index appearing twice in one term is regarded as the summation index. Using diadic representation, the operations of the tensor algebra reduce to those upon the basis vectors:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{(scalar product)} \tag{A2}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{kij} \mathbf{e}_k \quad \text{(vector product).} \tag{A3}$$

Here, ε_{ijk} are the components of the Levi–Civita tensor density, defined as follows: ε_{ijk} is skew-symmetric in its three indices; therefore, all those components which have at least two equal indices, vanish. The values of the nonvanishing components are ± 1, the sign depending on whether (i, j, k) is an even or odd permutation of $(1, 2, 3)$.

The identities eqn (13) can be readily obtained using diadic technique. The first identity of eqn (13) takes the form

$$\frac{\partial(\boldsymbol{\lambda} \times \dot{\mathbf{r}})}{\partial \dot{\mathbf{r}}} = \frac{\partial(\varepsilon_{imn} \lambda_m \dot{x}_n)}{\partial \dot{x}_j} \mathbf{e}_i \otimes \mathbf{e}_j = \varepsilon_{imn} \lambda_m \delta_{nj} \mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{\lambda} \times \hat{\mathbf{I}}. \tag{A4}$$

The second identity of eqn (13) is written as

$$\frac{\partial}{\partial \dot{\mathbf{r}}}\left(\frac{1}{|\dot{\mathbf{r}}|^{\alpha}}\right) = \frac{\partial}{\partial \dot{x}_i} \mathbf{e}_i (\dot{x}_m \dot{x}_m)^{-\alpha/2} = -\frac{\alpha}{2|\dot{\mathbf{r}}|^{\alpha+2}} \left(\frac{\partial \dot{x}_m}{\partial \dot{x}_i} \dot{x}_m + \dot{x}_m \frac{\partial \dot{x}_m}{\partial \dot{x}_i}\right) \mathbf{e}_i = -\frac{\alpha}{2|\dot{\mathbf{r}}|^{\alpha+2}} 2\delta_{mi} \dot{x}_m \mathbf{e}_i$$
$$= -\frac{\alpha}{|\dot{\mathbf{r}}|^{\alpha+2}} \dot{x}_i \mathbf{e}_i = -\alpha \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^{\alpha+2}}. \tag{A5}$$

Finally, the third identity of eqn (13) which will be extensively used in the forthcoming derivations is obtained as

$$\frac{\partial}{\partial \dot{\mathbf{r}}}\left(\frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha}}\right) = \frac{\partial}{\partial \dot{x}_i} (\boldsymbol{\varepsilon}_{mnp} \lambda_n \dot{x}_p \mathbf{e}_m \cdot \boldsymbol{\varepsilon}_{rst} \lambda_s \dot{x}_t \mathbf{e}_r)^{-\alpha/2} \quad \mathbf{e}_i = -\frac{\alpha}{2|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+2}} \frac{\partial}{\partial \dot{x}_i} (\boldsymbol{\varepsilon}_{mnp} \boldsymbol{\varepsilon}_{mst} \lambda_n \lambda_s \dot{x}_p \dot{x}_t) \mathbf{e}_i$$
$$= -\frac{\alpha}{2|\boldsymbol{\lambda} + \dot{\mathbf{r}}|^{\alpha+2}} \boldsymbol{\varepsilon}_{mnp} \boldsymbol{\varepsilon}_{mst} \lambda_n \lambda_s (\delta_{pi} \dot{x}_t + \delta_{ti} \dot{x}_p) \mathbf{e}_i = -\frac{\alpha(\boldsymbol{\varepsilon}_{mni} \boldsymbol{\varepsilon}_{mst} \lambda_n \lambda_s \dot{x}_t + \boldsymbol{\varepsilon}_{mnp} \boldsymbol{\varepsilon}_{msi} \lambda_n \lambda_s \dot{x}_p)}{2|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+2}} \mathbf{e}_i$$
$$= -\frac{\alpha \boldsymbol{\varepsilon}_{mni} \boldsymbol{\varepsilon}_{mst} \lambda_n \lambda_s \dot{x}_t}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+2}} \mathbf{e}_i = -\alpha \frac{\boldsymbol{\varepsilon}_{imn} (\boldsymbol{\varepsilon}_{mst} \lambda_s \dot{x}_t) \lambda_n}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+2}} \mathbf{e}_i = -\alpha \frac{(\boldsymbol{\lambda} \times \dot{\mathbf{r}}) \times \boldsymbol{\lambda}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+2}} = \alpha \frac{\boldsymbol{\lambda} \times \mathbf{e}'_2}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{\alpha+1}}. \tag{A6}$$

Coefficients of the expansion of the basis vector $\mathbf{e}'_1$

Differentiation of the first basis vector of eqn (7) with respect to $\dot{\mathbf{r}}$ yields

$$\frac{\partial \mathbf{e}'_1}{\partial \dot{\mathbf{r}}} = \frac{\partial}{\partial \dot{x}_j}\left(\frac{\dot{x}_i}{|\dot{\mathbf{r}}|}\right) \mathbf{e}_i \otimes \mathbf{e}_j = \left(\frac{1}{|\dot{\mathbf{r}}|} \frac{\partial \dot{x}_i}{\partial \dot{x}_j} + \dot{x}_i \frac{\partial |\dot{\mathbf{r}}|}{\partial \dot{x}_j}\right) \mathbf{e}_i \otimes \mathbf{e}_j = \left(\frac{\delta_{ij}}{|\dot{\mathbf{r}}|} - \frac{\dot{x}_i \dot{x}_j}{|\dot{\mathbf{r}}|^3}\right) \mathbf{e}_i \otimes \mathbf{e}_j = \frac{\hat{\mathbf{I}}}{|\dot{\mathbf{r}}|} - \frac{\dot{\mathbf{r}} \otimes \dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^3}. \tag{A7}$$

Repeated differentiation of the relation eqn (A7) results in

$$\frac{\partial^2 \mathbf{e}'_1}{\partial \dot{\mathbf{r}}\, \partial \dot{\mathbf{r}}} = \frac{\partial}{\partial \dot{x}_k}\left(\frac{\delta_{ij}}{|\dot{\mathbf{r}}|} - \frac{\dot{x}_i \dot{x}_j}{|\dot{\mathbf{r}}|^3}\right)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \left(\delta_{ij}\frac{\partial |\dot{\mathbf{r}}|^{-1}}{\partial \dot{x}_k} - \frac{1}{|\dot{\mathbf{r}}|^3}\frac{\partial(\dot{x}_i\dot{x}_j)}{\partial \dot{x}_k} - \dot{x}_i\dot{x}_j\frac{\partial|\dot{\mathbf{r}}|^{-3}}{\partial \dot{x}_k}\right)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

$$= \left(-\frac{\delta_{ij}\dot{x}_k + \dot{x}_i\delta_{jk} + \dot{x}_j\delta_{ik}}{|\dot{\mathbf{r}}|^3} + \frac{3\dot{x}_i\dot{x}_j\dot{x}_k}{|\dot{\mathbf{r}}|^5}\right)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = -\frac{\hat{\mathbf{I}} \otimes \dot{\mathbf{r}} + \dot{\mathbf{r}} \otimes \hat{\mathbf{I}} + (\dot{\mathbf{r}} \otimes \hat{\mathbf{I}})^T}{|\dot{\mathbf{r}}|^3} + \frac{3\dot{\mathbf{r}} \otimes \dot{\mathbf{r}} \otimes \dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^5}. \quad \text{(A8)}$$

Evaluation of the eqns (A7) and (A8) at $\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle$ yields coefficients $\hat{\mathbf{b}}^{(1)}$ and $\hat{\mathbf{c}}^{(1)}$ in eqns (11) and (12), respectively.

Coefficients of the expansion of the basis vector $\mathbf{e}'_2$

Employing the relations eqns (A4) and (A6), the derivative of the second basis vector with respect to $\dot{\mathbf{r}}$ is derived as

$$\frac{\partial \mathbf{e}'_2}{\partial \dot{\mathbf{r}}} = \frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|}\frac{\partial(\boldsymbol{\lambda} \times \dot{\mathbf{r}})}{\partial \dot{\mathbf{r}}} + \boldsymbol{\lambda} \times \dot{\mathbf{r}} \otimes \frac{\partial|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|}{\partial \dot{\mathbf{r}}} = \frac{\boldsymbol{\lambda} \times \hat{\mathbf{I}}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|} - \frac{(\boldsymbol{\lambda} \times \dot{\mathbf{r}}) \otimes (\boldsymbol{\lambda} \times \dot{\mathbf{r}} \times \boldsymbol{\lambda})}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^3}$$

$$= \frac{\boldsymbol{\lambda} \times \hat{\mathbf{I}} - \mathbf{e}'_2 \otimes (\mathbf{e}'_2 \times \boldsymbol{\lambda})}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|}. \quad \text{(A9)}$$

The coordinate form of the eqn (A9) is

$$\frac{\partial \mathbf{e}'_2}{\partial \dot{\mathbf{r}}} = \left(\frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|}\varepsilon_{imn}\lambda_m\delta_{nj} - \frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^3}(\varepsilon_{ipl}\lambda_p\dot{x}_l)\varepsilon_{jmn}(\varepsilon_{mst}\lambda_s\dot{x}_t)\lambda_s\right)\mathbf{e}_i \otimes \mathbf{e}_j. \quad \text{(A10)}$$

By differentiating both sides of eqn (A10), one obtains

$$\frac{\partial^2 \mathbf{e}'_2}{\partial \dot{\mathbf{r}}\, \partial \dot{\mathbf{r}}} = \boldsymbol{\lambda} \times \hat{\mathbf{I}} \otimes \frac{\partial|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{-1}}{\partial \dot{\mathbf{r}}} - (\boldsymbol{\lambda} \times \dot{\mathbf{r}}) \otimes (\boldsymbol{\lambda} \times \dot{\mathbf{r}} \times \boldsymbol{\lambda}) \otimes \frac{\partial|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^{-3}}{\partial \dot{\mathbf{r}}} - \frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^3}(\varepsilon_{ipl}\lambda_p\delta_{lk})\varepsilon_{jmn}(\varepsilon_{mst}\lambda_s\dot{x}_t)\lambda_n\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

$$- \frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^3}(\varepsilon_{ipl}\lambda_p\dot{x}_l)\varepsilon_{jmn}(\varepsilon_{mst}\lambda_s\delta_{tk})\lambda_n\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \quad \text{(A11)}$$

The first, second, third, and fourth terms in the right hand side of eqn (A11) are denoted by Z_1, Z_2, Z_3, and Z_4, respectively. The first two terms, Z_1 and Z_2, are derived using eqns (A6) and the definition of the basis vector $\mathbf{e}'_2$, see eqn (7):

$$Z_1 = \boldsymbol{\lambda} \times \hat{\mathbf{I}} \otimes \frac{\partial(\boldsymbol{\lambda} \times \dot{\mathbf{r}})}{\partial \dot{\mathbf{r}}} = \boldsymbol{\lambda} \times \hat{\mathbf{I}} \otimes \frac{\boldsymbol{\lambda} \times \mathbf{e}'_2}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2} \quad \text{(A12)}$$

$$Z_2 = -(\boldsymbol{\lambda} \times \dot{\mathbf{r}}) \otimes (\boldsymbol{\lambda} \times \dot{\mathbf{r}} \times \boldsymbol{\lambda}) \otimes \frac{3\boldsymbol{\lambda} \times \mathbf{e}'_2}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^4} = 3\frac{\mathbf{e}'_2 \otimes (\boldsymbol{\lambda} \times \mathbf{e}'_2) \otimes (\boldsymbol{\lambda} \times \mathbf{e}'_2)}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}. \quad \text{(A13)}$$

After introducing notations $\hat{\mathbf{s}} = \boldsymbol{\lambda} \times \mathbf{e}'_2$ and $\hat{\mathbf{p}} = \boldsymbol{\lambda} \times \hat{\mathbf{I}}$, the term Z_3 is written as

$$Z_3 = -\frac{\varepsilon_{ipl}\lambda_p\delta_{lk}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\varepsilon_{jmn}\frac{\varepsilon_{mst}\lambda_s\dot{x}_t}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|}\lambda_n\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = -\frac{p_{ik}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}(\varepsilon_{jmn}e'_{2m}\lambda_n)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

$$= \frac{p_{ik}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}(\varepsilon_{jnm}\lambda_n e'_{2m})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \frac{p_{ik}\hat{s}_j}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \frac{(\hat{\mathbf{s}} \otimes \hat{\mathbf{p}})^T}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}$$

$$= \frac{((\boldsymbol{\lambda} \times \mathbf{e}'_2) \otimes (\boldsymbol{\lambda} \times \hat{\mathbf{I}}))^T}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}. \quad \text{(A14)}$$

Taking into account the identity $\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$ and recalling that $\boldsymbol{\lambda}$ is a unit vector, the fourth term in eqn (A11) is derived as

$$Z_4 = -\frac{1}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\frac{\varepsilon_{ipl}\lambda_p\dot{x}_l}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|}\varepsilon_{jmn}\varepsilon_{mst}\delta_{tk}\lambda_s\lambda_n\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \frac{e'_{2i}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\varepsilon_{jmn}\varepsilon_{msk}\lambda_s\lambda_n\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

$$= \frac{e'_{2i}}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\varepsilon_{jnm}\varepsilon_{skm}\lambda_s\lambda_n\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \frac{e'_{2i}(\delta_{js}\delta_{nk} - \delta_{jk}\delta_{ns})\lambda_s\lambda_n}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

$$= \frac{e'_{2i}(\lambda_j\lambda_k - \delta_{jk}\lambda_s\lambda_s)}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \frac{\mathbf{e}'_2 \otimes (\boldsymbol{\lambda} \otimes \boldsymbol{\lambda} - \hat{\mathbf{I}})}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}. \quad \text{(A15)}$$

Substitution of eqns (A12)–(15) into eqn (A11) results in

$$\frac{\partial^2 \mathbf{e}'_2}{\partial \dot{\mathbf{r}}\, \partial \dot{\mathbf{r}}} = \frac{\boldsymbol{\lambda} \times \hat{\mathbf{I}} \otimes \boldsymbol{\lambda} \times \mathbf{e}'_2 + 3\mathbf{e}'_2 \otimes (\boldsymbol{\lambda} \times \mathbf{e}'_2) \otimes (\boldsymbol{\lambda} \times \mathbf{e}'_2) + (\boldsymbol{\lambda} \times \mathbf{e}'_2 \otimes \boldsymbol{\lambda} \times \hat{\mathbf{I}})^T + \mathbf{e}'_2 \otimes (\boldsymbol{\lambda} \otimes \boldsymbol{\lambda} - \hat{\mathbf{I}})}{|\boldsymbol{\lambda} \times \dot{\mathbf{r}}|^2}. \tag{A16}$$

Equations (A9) and (A16) evaluated at $\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle$ provide coefficients $\hat{\mathbf{b}}^{(2)}$ and $\hat{\mathbf{c}}^{(2)}$ entering in eqns (11) and (12), respectively.

Coefficients of the expansion of the basis vector $\mathbf{e}'_3$

Recalling definition of the coefficients in the expansion eqn (9), namely,

$$\mathbf{a}^{(m)} = \mathbf{e}'_m \big|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} = a_i^{(m)} \mathbf{e}_i, \quad \hat{\mathbf{b}}^{(m)} = \frac{\partial \mathbf{e}'_m}{\partial \dot{\mathbf{r}}}\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} = b_{ij}^{(m)} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \hat{\mathbf{c}}^{(m)} = \frac{\partial^2 \mathbf{e}'_m}{\partial \dot{\mathbf{r}}\, \partial \dot{\mathbf{r}}}\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} = c_{ijk}^{(m)} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{A17}$$

the expansion coefficients of the third basis vector are evaluated as

$$\begin{aligned}
\hat{\mathbf{b}}^{(3)} &= \frac{\partial \mathbf{e}'_3}{\partial \dot{\mathbf{r}}}\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} = \frac{\partial(\varepsilon_{imn} e'_{1m} e'_{2n})}{\partial \dot{x}_j}\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} \mathbf{e}'_i \otimes \mathbf{e}'_j = \left(\varepsilon_{imn} \frac{\partial e'_{1m}}{\partial \dot{x}_j} e'_{2n} + \varepsilon_{imn} e'_{1m} \frac{\partial e'_{2n}}{\partial \dot{x}_j} \right)\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} \mathbf{e}'_i \otimes \mathbf{e}'_j \\
&= (\varepsilon_{imn} b_{mj}^{(1)} a_n^{(2)} + \varepsilon_{imn} a_m^{(1)} b_{nj}^{(2)}) \mathbf{e}'_i \otimes \mathbf{e}'_j = (-\varepsilon_{inm} a_n^{(2)} b_{mj}^{(1)} + \varepsilon_{imn} a_m^{(1)} b_{nj}^{(2)}) \mathbf{e}'_i \otimes \mathbf{e}' \\
&= \mathbf{a}^{(1)} \times \hat{\mathbf{b}}^{(2)} - \mathbf{a}^{(2)} \times \hat{\mathbf{b}}^{(1)}
\end{aligned} \tag{A18}$$

and

$$\begin{aligned}
\hat{\mathbf{c}}^{(3)} &= \frac{\partial^2 \mathbf{e}'_3}{\partial \dot{\mathbf{r}}\, \partial \dot{\mathbf{r}}}\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} = \frac{\partial^2 (\varepsilon_{imn} e'_{1m} e'_{2n})}{\partial \dot{x}_j\, \partial \dot{x}_k}\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} \mathbf{e}'_i \otimes \mathbf{e}'_j \otimes \mathbf{e}'_k \\
&= \varepsilon_{imn} \left(\frac{\partial^2 e'_{1m}}{\partial \dot{x}_j\, \partial \dot{x}_k} e'_{2n} + \frac{\partial e'_{1m}}{\partial \dot{x}_j} \frac{\partial e'_{2n}}{\partial \dot{x}_k} + \frac{\partial e'_{2n}}{\partial \dot{x}_j} \frac{\partial e'_{1m}}{\partial \dot{x}_k} + e'_{1m} \frac{\partial^2 e'_{2n}}{\partial \dot{x}_j\, \partial \dot{x}_k} \right)\bigg|_{\dot{\mathbf{r}} = \langle \dot{\mathbf{r}} \rangle} \mathbf{e}'_i \otimes \mathbf{e}'_j \otimes \mathbf{e}'_k \\
&= \varepsilon_{imn} (c_{mjk}^{(1)} a_n^{(2)} + b_{mj}^{(1)} b_{nk}^{(2)} + b_{nj}^{(2)} b_{mk}^{(1)} + a_m^{(1)} c_{njk}^{(2)}) \mathbf{e}'_i \otimes \mathbf{e}'_j \otimes \mathbf{e}'_k \\
&= (-\varepsilon_{inm} a_n^{(2)} c_{mjk}^{(1)} + \varepsilon_{imn} b_{mj}^{(1)} b_{nk}^{(2)} - \varepsilon_{inm} b_{nj}^{(2)} b_{mk}^{(1)} + \varepsilon_{imn} a_m^{(1)} c_{njk}^{(2)}) \mathbf{e}'_i \otimes \mathbf{e}'_j \otimes \mathbf{e}'_k \\
&= -\mathbf{a}^{(2)} \hat{\mathbf{c}}^{(1)} + (\hat{\mathbf{b}}^{(1)T} \times \hat{\mathbf{b}}^{(2)})^T - (\hat{\mathbf{b}}^{(2)T} \times \hat{\mathbf{b}}^{(1)})^T + \mathbf{a}^{(1)} \times \hat{\mathbf{c}}^{(2)}.
\end{aligned} \tag{A19}$$